EXAMINATION

27 April 2010 (am)

Subject CT1 — Financial Mathematics Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 (i) Explain the difference

(a) between options and futures
(b) between call options and put options

[4]

A security is priced at £60. Coupons are paid half-yearly. The next coupon is due in two months' time and will be £2.80. The risk-free force of interest is 6% per annum.

(ii) Calculate the forward price an investor should agree to pay for the security in three months' time assuming no arbitrage. [3]

[Total 7]

2 In January 2008, the government of a country issued an index-linked bond with a term of two years. Coupons were payable half-yearly in arrear, and the annual nominal coupon rate was 4%. Interest and capital payments were indexed by reference to the value of an inflation index with a time lag of six months.

A tax-exempt investor purchased £100,000 nominal at issue and held it to redemption. The issue price was £98 per £100 nominal.

The inflation index was as follows:

Date	*Inflation Index*
July 2007	110.5
January 2008	112.1
July 2008	115.7
January 2009	119.1
July 2009	123.2

(i) Calculate the investor's cashflows from this investment and state the month when each cashflow occurs. [3]

(ii) Calculate the annual effective money yield obtained by the investor to the nearest 0.1% per annum. [3]

[Total 6]

3 A company issues ordinary shares to an investor who is subject to income tax at 20%.

Under the terms of the ordinary share issue, the investor is to purchase 1,000,000 shares at a purchase price of 45p each on 1 January 2011.

No dividend is expected to be paid for 2 years. The first dividend payable on 1 January 2013 is expected to be 5p per share. Dividends will then be paid every 6 months in perpetuity. The two dividend payments in any calendar year are expected to be the same, but the dividend payment is expected to increase at the end of each year at a rate of 3% per annum compound.

Calculate the net present value of the investment on 1 January 2011 at an effective rate of interest of 8% per annum. [5]

4 An investor is considering purchasing a fixed interest bond at issue which pays half-yearly coupons at a rate of 6% per annum. The bond will be redeemed at £105 per £100 nominal in 10 years' time. The investor is subject to income tax at 20% and capital gains tax at 25%.

The inflation rate is assumed to be constant at 2.8571% per annum.

Calculate the price per £100 nominal if the investor is to obtain a net real yield of 5% per annum. [7]

5 Let f_t denote the one-year forward rate of interest over the year from time t to time $(t+1)$.

The current forward rates in the market are:

time, t	0	1	2	3
one-year forward rate, f_t	4.4% p.a.	4.7% p.a.	4.9% p.a.	5.0% p.a.

A fixed-interest security pays coupons annually in arrear at the rate of 7% per annum and is redeemable at par in exactly four years.

(i) Calculate the price per £100 nominal of the security assuming no arbitrage. [3]

(ii) Calculate the gross redemption yield of the security. [3]

(iii) Explain, without doing any further calculations, how your answer to part (ii) would change if the annual coupon rate on the security were 9% per annum (rather than 7% per annum). [2]

[Total 8]

6 The annual returns, i, on a fund are independent and identically distributed. Each year, the distribution of $1 + i$ is lognormal with parameters $\mu = 0.05$ and $\sigma^2 = 0.004$, where i denotes the annual return on the fund.

(i) Calculate the expected accumulation in 25 years' time if £3,000 is invested in the fund at the beginning of each of the next 25 years. [5]

(ii) Calculate the probability that the accumulation of a single investment of £1 will be greater than its expected value 20 years later. [5]

[Total 10]

7 A pension fund has to pay out benefits at the end of each of the next 40 years. The benefits payable at the end of the first year total £1 million. Thereafter, the benefits are expected to increase at a fixed rate of 3.8835% per annum compound.

(i) Calculate the discounted mean term of the liabilities using a rate of interest of 7% per annum effective. [5]

The pension fund can invest in both coupon-paying and zero-coupon bonds with a range of terms to redemption. The longest-dated bond currently available in the market is a zero-coupon bond redeemed in exactly 15 years.

(ii) Explain why it will not be possible to immunise this pension fund against small changes in the rate of interest. [2]

(iii) Describe the other practical problems for an institutional investor who is attempting to implement an immunisation strategy. [3]

[Total 10]

8 A loan is repayable by annual instalments paid in arrear for 20 years. The first instalment is £4,650 and each subsequent instalment is £150 greater than the previous instalment.

Calculate the following, using an interest rate of 9% per annum effective:

(i) the amount of the original loan [3]
(ii) the capital repayment in the tenth instalment [4]
(iii) the interest element in the last instalment [2]
(iv) the total interest paid over the whole 20 years [2]

[Total 11]

9 A company is undertaking a new project. The project requires an investment of £5m at the outset, followed by £3m three months later.

It is expected that the investment will provide income over a 15 year period starting from the beginning of the third year. Net income from the project will be received continuously at a rate of £1.7m per annum. At the end of this 15 year period there will be no further income from the investment.

Calculate at an effective rate of interest of 10% per annum:

(i) the net present value of the project [3]
(ii) the discounted payback period [4]

A bank has offered to loan the funds required to the company at an effective rate of interest of 10% per annum. Funds will be drawn from the bank when required and the loan can be repaid at any time. Once the loan is paid off, the company can earn interest on funds from the venture at an effective rate of interest of 7% per annum.

(iii) Calculate the accumulated profit at the end of the 17 years. [4]
[Total 11]

10 A pension fund's assets were invested with two fund managers.

On 1 January 2007 Manager A was given £120,000 and Manager B was given £100,000. A further £10,000 was invested with each manager on 1 January 2008 and again on 1 January 2009.

The values of the funds were:

	31 December 2007	*31 December 2008*	*31 December 2009*
Manager A	£130,000	£135,000	£180,000
Manager B	£140,000	£145,000	£150,000

(i) Calculate the time-weighted rates of return earned by Manager A and Manager B over the period 1 January 2007 to 31 December 2009. [4]

(ii) Show that the money-weighted rate of return earned by Manager A over the period 1 January 2007 to 31 December 2009 is approximately 9.4% per annum. [2]

(iii) Explain, without performing further calculations, whether the money-weighted rate of return earned by Manager B over the period 1 January 2007 to 31 December 2009 was higher than, lower than or equal to that earned by Manager A. [3]

(iv) Discuss the relative performance of the two fund managers. [3]
[Total 12]

11 The force of interest $\delta(t)$ is a function of time and at any time t, measured in years, is given by the formula

$$\delta(t) = \begin{cases} 0.04 + 0.02t & 0 \le t < 5 \\ 0.05 & 5 \le t \end{cases}.$$

(i) Derive and simplify as far as possible expressions for $v(t)$, where for $v(t)$ is the present value of a unit sum of money due at time t. [5]

(ii) (a) Calculate the present value of £1000 due at the end of 17 years.

(b) Calculate the rate of interest per annum convertible monthly implied by the transaction in part (ii)(a). [4]

A continuous payment stream is received at a rate of $10e^{0.01t}$ units per annum between $t = 6$ and $t = 10$.

(iii) Calculate the present value of the payment stream. [4]

[Total 13]

END OF PAPER

EXAMINERS' REPORT

April 2010 Examinations

Subject CT1 — Financial Mathematics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

July 2010

Comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Well-prepared candidates scored well across the whole paper and the examiners were pleased with the general standard of answers. However, questions that required an element of explanation or analysis were less well answered than those which just involved calculation. The comments below concentrate on areas where candidates could have improved their performance.

Q2.

A common error was to divide the nominal payments by the increase in the index factor (rather than multiplying).

Q3.

Many candidates made calculation errors in this question but may have scored more marks if their working had been clearer.

Q6.

Many candidates assumed that the accumulation in part (i) was for a single payment.

Q7.

The calculation was often performed well. In part (ii), many explanations were unclear and some candidates seemed confused between DMT and convexity although a correct explanation could involve either of these concepts.

Q9.

A common error was to assume that income only started after three years rather than 'starting from the beginning of the third year'.

Q10.

This question was answered well but examiners were surprised by the large number of candidates who used interpolation or other trial and error methods in part (ii) when the answer had been given in the question. The examiners recommend that students pay attention to the details given in the solutions to parts (iii) and (iv). For such questions, candidates should be looking critically at the figures given/calculated and making points specific to the scenario rather than just making general statements taken from the Core Reading.

1 (i) (a) Options – holder has the right but not the obligation to trade
Futures – both parties have agreed to the trade and are obliged to do so.

(b) Call Option – right but not the obligation to BUY specified asset at specified price at specified future date.

Put Option – right but not the obligation to SELL specified asset at specified price at specified future date.

(ii) $K = 60e^{0.06\times 3/12} - 2.80e^{0.06\times 1/12} = 60.90678 - 2.81404 = £58.09$

2 (i) Cash flows:

Issue price:	Jan 08	$-0.98\times 100,000$	$= -£98,000$
Interest payments:	July 08	$0.02\times 100,000\times\dfrac{112.1}{110.5}$	$= £2,028.96$
	Jan 09	$0.02\times 100,000\times\dfrac{115.7}{110.5}$	$= £2,094.12$
	July 09	$0.02\times 100,000\times\dfrac{119.1}{110.5}$	$= £2,155.66$
	Jan 10	$0.02\times 100,000\times\dfrac{123.2}{110.5}$	$= £2,229.86$
Capital redeemed:	Jan 10	$100,000\times\dfrac{123.2}{110.5}$	$= £111,493.21$

(ii) Equation of value is:

$$98000 = 2028.96v^{\frac{1}{2}} + 2094.12v + 2155.66v^{1\frac{1}{2}} + 2229.86v^2 + 111493.21v^2$$

At 11%, RHS = 97955.85 ≈ 98000

3 Purchase price = 0.45 × 1,000,000 = £450,000

PV of dividends = $50000\times(1-0.2)\times\left[\left(v^2+v^{2\frac{1}{2}}\right)+1.03\left(v^3+v^{3\frac{1}{2}}\right)+1.03^2\left(v^4+v^{4\frac{1}{2}}\right)+\cdots\right]$

$$= 40000\left(v^2+v^{2\frac{1}{2}}\right)\left[1+1.03v+1.03^2v^2+\cdots\right] \text{ @ } 8\%$$

$$= 40000\times1.68231\times\left(\frac{1}{1-1.03/1.08}\right)=1,453,516$$

⇒ NPV = 1,453,516 – 450,000 = £1,003,516

4 Let i = money yield

$\Rightarrow 1+i=1.0285714\times1.05=1.08 \quad \Rightarrow i=8\%$ p.a.

Check whether CGT is payable: compare $i^{(2)}$ with $(1-t)g$

$$(1-t)g=0.8\times\frac{6}{105}=0.04571$$

From tables, $i^{(2)}=7.8461\%$ $\Rightarrow i^{(2)}>(1-t)g$

⇒ CGT is payable

$P = 0.8\times6a^{(2)}_{\overline{10|}}+105v^{10}-0.25(105-P)v^{10}$ @ 8%

$$=\frac{0.8\times6a^{(2)}_{\overline{10|}}+0.75\times105v^{10}}{1-0.25v^{10}}$$

$$=\frac{4.8\times1.019615\times6.7101+78.75\times0.46319}{1-0.25\times0.46319}$$

= £78.39

5 (i) Let P denote the current price (per £100 nominal) of the security.

Then, we have:

$$P = \frac{7}{1.044} + \frac{7}{1.044 \times 1.047} + \frac{7}{1.044 \times 1.047 \times 1.049} + \frac{107}{1.044 \times 1.047 \times 1.049 \times 1.05} = 108.0872$$

(ii) The gross redemption yield, i, is given by:

$$108.09 = 7 \times a_{\overline{4}|}^{i\%} + 100 \times v_{i\%}^{4}$$

Then, we have:

$$\left.\begin{array}{lll} i = 5\% & \Rightarrow & RHS = 107.0919 \\ i = 4.5\% & \Rightarrow & RHS = 108.9688 \end{array}\right\} \Rightarrow i \approx 0.045 + (0.05 - 0.045) \times \left(\frac{108.0872 - 108.9688}{107.0919 - 108.9688} \right) = 0.0473$$

(iii) The gross redemption yield represents a weighted average of the forward rates at each duration, weighted by the cash flow received at that time.

Thus, increasing the coupon rate will increase the weight applied to the cash flows at the early durations and, as the forward rates are lower at early durations, the gross redemption yield on a security with a higher coupon rate will be lower than above.

Note to markers: no marks for simply plugging 9% pa in, and providing no explanation for result.

6 (i) $E(1+i) = e^{\mu + \frac{1}{2}\sigma^2}$

$$= e^{0.05 + \frac{1}{2} \times 0.004}$$

$$= 1.0533757$$

$\therefore E[i] = 0.0533757$ since $E(1+i) = 1 + E(i)$

Let A be the accumulation at the end of 25 years of £3,000 paid annually in advance for 25 years.

Then $E\left[A\right] = 3000\ddot{s}_{\overline{25|}}$ at rate $j = 0.0533757$

$$= 3000\frac{\left((1+j)^{25}-1\right)}{j}\times(1+j)$$

$$= 3000\frac{\left(1.0533757^{25}-1\right)}{0.0533757}\times 1.0533757$$

$$= £158,036.43$$

(ii) Let the accumulation be S_{20}

S_{20} has a log-normal distribution with parameters 20μ and $20\sigma^2$

$\therefore E\left[S_{20}\right] = e^{20\mu + \frac{1}{2}\times 20\sigma^2}$

$\left\{\text{or}\,(1+j)^{20}\right\}$

$= \exp\left(20\times 0.05 + 10\times 0.004\right)$

$= e^{1.04} = 2.829217$

In $S_{20} \sim N\left(20\mu, 20\sigma^2\right)$

$\Rightarrow$ In $S_{20} \sim N\left(1, 0.08\right)$

$\Pr\left(S_{20} > 2.829217\right) = \Pr\left(\ln S_{20} > \ln 2.829217\right)$

$$= \Pr\left(Z > \frac{\ln 2.829217\text{-}1}{\sqrt{0.08}}\right) \quad \text{where} \quad Z \sim N(0,1)$$

$$= \Pr\left(Z > 0.14\right) = 1 - \Phi(0.14)$$

$= 1 – 0.55567$

$= 0.44433$ i.e. 44.4%

7 (i) DMT of liabilities is given by:

$$\frac{1\times 1\times v_{7\%} + 2\times(1.038835)\times v_{7\%}^2 + 3\times(1.038835)^2\times v_{7\%}^3 + \ldots + 40\times(1.038835)^{39}\times v_{7\%}^{40}}{1\times v_{7\%} + (1.038835)\times v_{7\%}^2 + (1.038835)^2\times v_{7\%}^3 + \ldots + (1.038835)^{39}\times v_{7\%}^{40}}$$

$$= \frac{(1.038835)^{-1}\times\left[\left(\frac{1.038835}{1.07}\right) + 2\times\left(\frac{1.038835}{1.07}\right)^2 + 3\times\left(\frac{1.038835}{1.07}\right)^3 + \ldots + 40\times\left(\frac{1.038835}{1.07}\right)^{40}\right]}{(1.038835)^{-1}\times\left[\left(\frac{1.038835}{1.07}\right) + \left(\frac{1.038835}{1.07}\right)^2 + \left(\frac{1.038835}{1.07}\right)^3 + \ldots + \left(\frac{1.038835}{1.07}\right)^{40}\right]}$$

$$= \frac{v_{i^*} + 2\times v_{i^*}^2 + 3\times v_{i^*}^3 + \ldots + 40\times v_{i^*}^{40}}{v_{i^*} + v_{i^*}^2 + v_{i^*}^3 + \ldots + v_{i^*}^{40}}$$

$$= \frac{(Ia)_{\overline{40|}}^{i^*}}{a_{\overline{40|}}^{i^*}}$$

where $v_{i^*} \equiv \frac{1}{1+i^*} = \frac{1.038835}{1.07} \Rightarrow i^* = \frac{1.07}{1.038835} - 1 = \frac{0.07-0.038835}{1.038835} = 0.03$.

Hence, DMT of liabilities is:

$$\frac{(Ia)_{\overline{40|}}^{3\%}}{a_{\overline{40|}}^{3\%}} = \frac{384.8647}{23.1148} = 16.65 \text{ years}$$

(Alternative method for DMT formula

$$DMT = \frac{v(1+2gv+3g^2v^2+\cdots+40g^{39}v^{39})}{v(1+gv+g^2v^2+\cdots+g^{39}v^{39})} = \frac{v(I\ddot{a})_{\overline{40|}}^{3\%}}{v\ddot{a}_{\overline{40|}}^{3\%}} = \frac{(I\ddot{a})_{\overline{40|}}^{3\%}}{\ddot{a}_{\overline{40|}}^{3\%}} = \frac{(Ia)_{\overline{40|}}^{3\%}}{a_{\overline{40|}}^{3\%}}$$

where $g = 1.038835$.)

(ii) Even if the fund manager invested entirely in the 15-year zero-coupon bond, the DMT of the assets will be only 15 years (and, indeed, any other portfolio of securities will result in a lower DMT).

Thus, it is not possible to satisfy the second condition required for immunisation (i.e. DMT of assets = DMT of liabilities).

Hence, the fund cannot be immunised against small changes in the rate of interest.

(iii) The other problems with implementing an immunisation strategy in practice include:

- the approach requires a continuous re-structuring of the asset portfolio to ensure that the volatility of the assets remains equal to that of the liabilities over time
- for most institutional investors, the amounts and timings of the cash flows in respect of the liabilities are unlikely to be known with certainty
- institutional investor is only immunised for small changes in the rate of interest
- the yield curve is unlikely to be flat at all durations
- changes in the term structure of interest rates will not necessarily be in the form of a parallel shift in the curve (e.g. the shape of the curve can also change from time to time)

8 (i) Loan = $4500a_{\overline{20|}} + 150\left(Ia\right)_{\overline{20|}}$ at 9%

$\Rightarrow$ Loan = $4500 \times 9.1285 + 150 \times 70.9055$

$= 41,078.25 + 10,635.83 = 51,714.08$

(ii) Loan o/s after 9th year = $\left(4500 + 1350\right)a_{\overline{11|}} + 150\left(Ia\right)_{\overline{11|}}$ at 9%

Loan o/s = $5,850 \times 6.8052 + 150 \times 35.0533$

$= 39,810.42 + 5258.00 = 45,068.42$

Repayment = $6000 - 45,068.42 \times 0.09 = £1,943.84$

(Alternative solution to (ii)

(ii) Loan o/s after 9th year = $\left(4500 + 1350\right)a_{\overline{11|}} + 150\left(Ia\right)_{\overline{11|}}$ at 9%

$= 5,850 \times 6.8052 + 150 \times 35.0533 = 45,068.42$ as before

Loan o/s after 10th year = $\left(4500 + 1500\right)a_{\overline{10|}} + 150\left(Ia\right)_{\overline{10|}}$ at 9%

$= 6,000 \times 6.4177 + 150 \times 30.7904 = 43,124.76$

Repayment = $45,068.42 - 43,124.76 = £1,943.66$)

(iii) Last instalment = $4650 + 19 \times 150 = 7500$

Loan o/s = $7500 a_{\overline{1|}} = 7500v$

Interest = $7500 \times 0.91743 \times 0.09 = £619.27$

(iv) Total payments $= 20 \times 4650 + \frac{1}{2} \times 19 \times 20 \times 150$

$= 93,000 + 28,500 = 121,500$

Total interest = 121,500 – 51,714.08 = £69,785.92

9 (i) NPV $= -5 - 3v^{1/4} + 1.7\bar{a}_{\overline{15|}} v^2$ @10%

NPV = $-5 - 3 \times 0.976454 + 1.7 \times 0.82645 \times \frac{i}{\delta} a_{\overline{15|}}$ @10%

$= -5 - 2.929362 + 1.404965 \times 1.049206 \times 7.6061$

$= -7.929362 + 11.21213458$

$= 3.282772575$

NPV = £3.283m

(ii) DPP is $t + 2$ such that

$1.7\bar{a}_{\overline{t|}} v^2 = 5 + 3v^{\frac{1}{4}} \Rightarrow 1.474097708 a_{\overline{t|}} = 7.929362$ @10%

$\frac{1 - 1.1^{-t}}{0.1} = 5.379129 \Rightarrow 1 - 1.1^{-t} = 0.5379129$

$\Rightarrow 0.4620871 = 1.1^{-t} \Rightarrow \ln 0.4620871 = -t \ln 1.1$

$\Rightarrow t = 8.100$

$\therefore DPP = 10.1$ years

(iii) Accumulated profit 17 years from start of project:

$$= 1.7 s_{\overline{6.9}|7\%} = 1.7 \times \frac{\left(1.07^{6.9} - 1\right)}{\delta} \quad @\, 7\%$$

$$= 1.7 \times \frac{\left(1.07^{6.9} - 1\right)}{0.067659}$$

$$= 1.7 \times 8.79346$$

$$= £14.95m$$

10 (i) The values of the funds before and after the cash injections are:

	Manager A		*Manager B*	
1 January 2007	120,000		100,000	
31 December 2007	130,000	140,000	140,000	150,000
31 December 2008	135,000	145,000	145,000	155,000
31 December 2009	180,000		150,000	

Thus, TWRR for Manager A is given by:

$$(1+i)^3 = \frac{130}{120} \times \frac{135}{140} \times \frac{180}{145} \Rightarrow i = 0.0905 \text{ or } 9.05\%$$

And, TWRR for Manager B is given by:

$$(1+i)^3 = \frac{140}{100} \times \frac{145}{150} \times \frac{150}{155} \Rightarrow i = 0.0941 \text{ or } 9.41\%$$

(ii) MWRR for Manager A is given by:

$$120 \times (1+i)^3 + 10 \times (1 \times i)^2 + 10 \times (1+i) = 180$$

Then, putting $i = 0.094$ gives $LHS = 180.03$ which is close enough to 180.

(iii) Both funds increased by 50% over the three year period and received the same cashflows at the same times.

Since the initial amount in fund B was lower, the cash inflows received represent a larger proportion of fund B and hence the money weighted return earned by fund B over the period will be lower, particularly since the returns were negative for the 2nd and 3rd years.

[Could also note that for fund B:

$$100\times(1+i)^3+10\times(1\times i)^2+10\times(1+i)=150$$

So by a proportional argument $120\times(1+i)^3+12\times(1\times i)^2+12\times(1+i)=180$

which when compared with the equation for fund A in (ii) clearly shows that the return for B is lower.]

(iv) The money weighted rate of return is higher for fund A, whilst the time weighted return is higher for fund B.

When comparing the performance of investment managers, the time weighted rate of return is generally better because it ignores the effects of cash inflows or outflows being made which are beyond the manager's control.

In this case, Manager A's best performance is in the final year, when the fund was at its largest, whilst Manager B's best performance was in the first year, where his fund was at its lowest.

Overall, it may be argued that Manager B has performed slightly better than Manager A since Manager B achieved the higher time weighted return.

11 (i) $t<5$

$$v(t) = e^{-\int_0^t (0.04+0.02s)ds}$$

$$= e^{-\left[0.04s+0.01s^2\right]_0^t}$$

$$= e^{-\left[0.04t+0.01t^2\right]}$$

$t\geq 5$

$$v(t) = e^{-\left\{\int_0^5 (0.04+0.02s)ds+\int_5^t 0.05ds\right\}}$$

$$= v(5)\times e^{-\left[0.05(t-5)\right]}$$

$$= e^{-0.45}\times e^{-\left[0.05(t-5)\right]} = e^{-\left[0.05t+0.2\right]}$$

(ii) (a) $PV = 1{,}000e^{-[0.05\times 17+0.2]} = e^{-1.05}$

$= 349.94$

(b) $1000\left(1+\dfrac{i^{(12)}}{12}\right)^{-204} = 349.94$

$\Rightarrow i^{(12)} = 6.1924\%$

(iii) $PV = \int_6^{10} e^{-0.45}e^{-[0.05t-0.25]}10e^{0.01t}\,dt$

$= 10e^{-0.2}\int_6^{10} e^{-0.04t}\,dt$

$= 10e^{-0.2}\left[-\dfrac{e^{-0.04t}}{0.04}\right]_6^{10}$

$= 8.18733\times 2.90769$

$= 23.806$

(Alternative Solution to (iii)

Accumulated value at time $t = 10$

$= \int_6^{10} 10e^{0.01t}\left(\exp\int_t^{10} 0.05ds\right)dt$

$= \int_6^{10} 10e^{0.01t}\left(\exp[0.05s]_t^{10}\right)dt$

$= \int_6^{10} 10e^{0.01t}e^{0.5-0.05t}dt = \int_6^{10} 10e^{0.5-0.04t}dt$

$= \left[\dfrac{10e^{0.5-0.04t}}{-0.04}\right]_6^{10} = -276.293 + 324.233 = 47.940$

Present value = $v(10)\times 47.940 = 0.63763e^{-[0.05\times 10-0.25]}\times 47.940 = 23.806$

END OF EXAMINERS' REPORT

EXAMINATION

7 October 2010 (am)

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Core Technical

Time allowed: Three hours

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AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 A bond pays coupons in perpetuity on 1 June and 1 December each year. The annual coupon rate is 3.5% per annum. An investor purchases a quantity of this bond on 20 August 2009.

Calculate the price per £100 nominal to provide the investor with an effective rate of return per annum of 10%. [3]

2 A bond is redeemed at £110 per £100 nominal in exactly four years' time. It pays coupons of 4% per annum half-yearly in arrear and the next coupon is due in exactly six months' time. The current price is £110 per £100 nominal.

(i) (a) Calculate the gross rate of return per annum convertible half-yearly from the bond.

(b) Calculate the gross effective rate of return per annum from the bond. [2]

(ii) Calculate the net effective rate of return per annum from the bond for an investor who pays income tax at 25%. [2]

[Total 4]

3 The annual rates of return from an asset are independently and identically distributed. The expected accumulation after 20 years of £1 invested in this asset is £2 and the standard deviation of the accumulation is £0.60.

(a) Calculate the expected effective rate of return per annum from the asset, showing all the steps in your working.

(b) Calculate the variance of the effective rate of return per annum.

[6]

4 A six-month forward contract was issued on 1 April 2009 on a share with a price of 700p at that date. It was known that a dividend of 20p per share would be paid on 1 May 2009. The one-month spot, risk-free rate of interest at the time of issue was 5% per annum effective and the forward rate of interest from 1 May to 30 September was 3% per annum effective.

(i) Calculate the forward price at issue, assuming no arbitrage, explaining your working. [3]

It has been suggested that the forward price cannot be calculated without making a judgement about the expected price of the share when the forward contract matures.

(ii) Explain why this statement is not correct. [2]

(iii) Comment on whether the method used in part (i) would still be valid if it was not known with certainty that the dividend due on 1 May 2009 would be paid.

[1]

[Total 6]

5 (a) Describe the characteristics of Eurobonds.

(b) Describe the characteristics of convertible bonds.

[6]

6 On 1 January 2001 the government of a particular country bought 200 million shares in a particular bank for a total price of £2,000 million. The shares paid no dividends for three years. On 30 June 2004 the shares paid dividends of 10 pence per share. On 31 December 2004, they paid dividends of 20 pence per share. Each year, until the end of 2009, the dividend payable every 30 June rose by 10% per annum compound and the dividend payable every 31 December rose by 10% per annum compound. On 1 January 2010, the shares were sold for their market price of £3,500 million.

(i) Calculate the net present value on 1 January 2001 of the government's investment in the bank at a rate of interest of 8% per annum effective. [5]

(ii) Calculate the accumulated profit from the government's investment in the bank on the date the shares are sold using a rate of interest of 8% per annum effective. [1]

[Total 6]

7 (i) State the three conditions that are necessary for a fund to be immunised from small, uniform changes in the rate of interest. [2]

(ii) A pension fund has liabilities of £10m to meet at the end of each of the next ten years. It is able to invest in two zero-coupon bonds with a term to redemption of three years and 12 years respectively. The rate of interest is 4% per annum effective.

Calculate:

(a) the present value of the liabilities of the pension fund

(b) the duration of the liabilities of the pension fund

(c) the nominal amount that should be invested in the zero-coupon bonds to ensure that the present values and durations of the assets and liabilities is the same

[7]

(iii) One year later, just before the pension payment then due, the rate of interest is 5% per annum effective.

(a) Determine whether the duration of the assets and the liabilities are still equal.

(b) Comment on the practical usefulness of the theory of immunisation in the context of the above result.

[6]

[Total 15]

8 The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula

$$\delta(t) = \begin{cases} 0.05 + 0.001t & 0 \le t \le 20 \\ 0.05 & t > 20 \end{cases}$$

(i) Derive and simplify as far as possible expressions for $v(t)$, where $v(t)$ is the present value of a unit sum of money due at time t. [5]

(ii) (a) Calculate the present value of £100 due at the end of 25 years.

(b) Calculate the rate of discount per annum convertible quarterly implied by the transaction in part (ii)(a). [4]

(iii) A continuous payment stream is received at rate $30e^{-0.015t}$ units per annum between $t = 20$ and $t = 25$. Calculate the accumulated value of the payment stream at time $t = 25$. [4]

[Total 13]

9 The government of a particular country has just issued three bonds with terms to redemption of exactly one, two and three years respectively. Each bond is redeemed at par and pays coupons of 8% annually in arrear. The annual effective gross redemption yields from the one, two and three year bonds are 4%, 3% and 3% respectively.

(i) Calculate the one-year, two-year and three-year spot rates of interest at the date of issue. [8]

(ii) Calculate all possible forward rates of interest from the above spot rates of interest. [4]

An index of retail prices has a current value of 100.

(iii) Calculate the expected level of the retail prices index in one year, two years' and three years' time if the expected real spot rates of interest are 2% per annum effective for all terms. [5]

(iv) Calculate the expected rate of inflation per annum in each of the next three years. [2]

[Total 19]

10 On 1 April 2003 a company issued securities that paid no interest and that were to be redeemed for £70 after five years. The issue price of the securities was £64. The securities were traded in the market and the market prices at various different dates are shown in the table below.

Date	*Market price of securities (£)*
1 April 2003	64
1 April 2004	65
1 April 2005	60
1 April 2006	65
1 April 2007	68
1 April 2008	70

(i) Explain why the price of the securities might have fallen between 1 April 2004 and 1 April 2005. [1]

Two investors bought the securities at various dates. Investor X bought 100 securities on 1 April 2003 and 1,000 securities on 1 April 2005. Investor Y bought 100 securities every year on 1 April from 2003 to 2007 inclusive. Both investors held the securities until maturity.

(ii) Construct a table showing the nominal amount of the securities held and the market value of the holdings for X and Y on 1 April each year, just before any purchases of securities. [5]

(iii) (a) Calculate the effective money weighted rate of return per annum for X for the period from 1 April 2003 to 1 April 2008.

(b) Calculate the effective time weighted rate of return per annum for X for the period from 1 April 2003 to 1 April 2008.

[6]

(iv) (a) Determine whether the effective money weighted rate of return for Y is lower or higher than that for X for the period from 1 April 2003 to 1 April 2008.

(b) Determine the effective time weighted rate of return per annum for Y for the period from 1 April 2003 to 1 April 2008.

[7]

(v) Discuss the relationship between the different rates of return that have been calculated. [3]

[Total 22]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2010 Examinations

Subject CT1 — Financial Mathematics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

December 2010

Comments

Please note that different answers may be obtained from those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown. Candidates also lose marks for not showing their working in a methodical manner which the examiner can follow. This can particularly affect candidates on the pass/fail borderline when the examiners have to make a judgement as to whether they can be sure that the candidate has communicated a sufficient command of the syllabus to be awarded a pass.

The general standard of answers was noticeably lower than in previous sessions and there were a significant number of very ill-prepared candidates. As in previous exams, questions that required an element of explanation or analysis were less well answered than those which just involved calculation.

Comments on individual questions, where relevant, can be found after the solution to each question. These comments concentrate on areas where candidates could have improved their performance.

1 Working in half years:

The present value of the security on 1st June would have been $\dfrac{3.5}{i^{(2)}}$

20 August is 80 days later so the present value is $\dfrac{3.5}{i^{(2)}}(1+i)^{80/365}$

Hence the price per £100 nominal is $\dfrac{3.5}{0.097618}(1.1)^{80/365} = £36.611$

2 (i) (a) Gross rate of return convertible half yearly is simply 4/110 = 0.03636 or 3.636%.

(b) Gross effective rate of return is $\left(1+\dfrac{0.03636}{2}\right)^2 - 1 = 0.03669$ or 3.669%

(ii) The net effective rate of return per half year is $0.75\times\dfrac{0.03636}{2} = 0.013635$.

The net effective rate of return per annum is therefore:

$(1.013635)^2 - 1 = 0.02746$ or 2.746%.

A common error was to divide the nominal payments by the increase in the index factor (rather than multiplying).

3 (a) Let S_{20} be the accumulation of the unit investment after 20 years:

$$E(S_{20}) = E\left[(1+i_1)(1+i_2)\ldots(1+i_{20})\right]$$

$$E(S_{20}) = E[1+i_1]E[1+i_2]\ldots E[1+i_{20}] \text{ as } \{i_t\} \text{ are independent}$$

$$E[i_t] = j \quad \therefore \quad E(S_{20}) = (1+j)^{20} = 2$$

$$\Rightarrow j = 2^{1/20} - 1 = 3.5265\%$$

(b) The variance of the effective rate of return per annum is s^2 where

$$\operatorname{Var}[S_n] = \left((1+j)^2 + s^2\right)^{20} - (1+j)^{40} = 0.6^2$$

$$s^2 = \left[0.6^2 + \left((1+j)^{20}\right)^2\right]^{1/20} - (1+j)^2$$

$$= \left(0.6^2 + 2^2\right)^{1/20} - 2^{1/10} = 0.004628$$

Many candidates made calculation errors in this question but may have scored more marks if their working had been clearer.

4 (i) Assuming no arbitrage, buying the share is the same as buying the forward except that the cash does not have to be paid today and a dividend will be payable from the share.

Therefore, price of forward is:

$$700(1.05)^{1/12}(1.03)^{5/12} - 20(1.03)^{5/12}$$
$= 711.562 - 20.248 = 691.314$

(ii) The no arbitrage assumption means that we can compare the forward with the asset from which the forward is derived and for which we know the market price. As such we can calculate the price of the forward from this, without knowing the expected price at the time of settlement. [It could also be mentioned that the market price of the underlying asset does, of course, already incorporate expectations].

(iii) If it was not known with certainty that the dividend would be received we could not use a risk-free interest rate to link the cash flows involved with the purchase of the forward with all the cash flows from the underlying asset.

5 (a) Eurobonds

- Medium-to-long-term borrowing.
- Pay regular coupon payments and a capital payment at maturity.
- Issued by large corporations, governments or supranational organisations.
- Yields to maturity depend on the risk of the issuer.
- Issued and traded internationally (not in core reading).
- Often have novel features.
- Usually unsecured
- Issued in any currency
- Normally large issue size
- Free from regulation of any one government

(b) Convertible Securities

- Generally unsecured loan stocks.
- Can be converted into ordinary shares of the issuing company.
- Pay interest/coupons until conversion.
- Provide levels of income between that of fixed-interest securities and equities.
- Risk characteristics vary as the final date for convertibility approaches.
- Generally less volatility than in the underlying share price before conversion.
- Combine lower risk of debt securities with the potential for gains from equity investment.
- Security and marketability depend upon issuer
- Generally provide higher income than ordinary shares and lower income than conventional loan stock or preference shares

6 (i) Net present value (all figures in £*m*)

$$= -2,000 + 0.1 \times 200 \times v^3 \left(v^{0.5} + 1.1v^{1.5} + 1.1^2 v^{2.5} + \ldots + 1.1^5 v^{5.5}\right)$$
$$+ 0.2 \times 200 \times v^3 \left(v + 1.1v^2 + 1.1^2 v^3 + \ldots + 1.1^5 v^6\right) + 3,500v^9$$

at 8% per annum effective.

$$= -2,000 + \frac{200}{1.1}\left(0.1v^{2.5} + 0.2v^3\right)\left(1.1v + (1.1v)^2 + (1.1v)^3 + \ldots + (1.1v)^6\right) + 3,500v^9$$

$$= -2,000 + \frac{200}{1.1}\left(0.1v^{2.5} + 0.2v^3\right)a'_{\overline{6}|} + 3,500v^9$$

where the annuity is evaluated at a rate of $\frac{0.08 - 0.1}{1 + 0.1} = -1.818\%$ per annum effective.

$$a'_{\overline{6}|} = \frac{1 - (1 - 0.018181)^{-6}}{-0.018181} = 6.4011$$

and so net present value is

$$-2,000 + \frac{200}{1.1}\left(0.1 \times 1.08^{-2.5} + 0.2 \times 1.08^{-3}\right) \times 6.4011 + 3,500 \times 1.08^{-9} = £31.66m$$

(ii) Accumulated profit at the time of sale is $31.66 \times 1.08^9 = £63.30m$

Many candidates assumed that the accumulation in part (i) was for a single payment.

7 (i) The present value of the assets is equal to the present value of the liabilities. The duration of the assets is equal to the duration of the liabilities.

The spread of the asset terms around the duration is greater than that for the liability terms (or, equivalently, convexity of assets is greater).

(ii) (a) Present value of liabilities (in £m)
$=10a_{\overline{10|}}$ at 4% $=10\times 8.1109=81.109$

(b) Duration is equal to $\dfrac{10(Ia)_{\overline{10|}}}{10a_{\overline{10|}}}$ at 4% $=\dfrac{41.9922}{8.1109}=5.1773$ years

(c) Let the amounts to be invested in the two zero coupon bonds be X and Y.

$$Xv^3+Yv^{12}=81.109 \qquad (1)$$

$$3Xv^3+12Yv^{12}=419.922 \qquad (2)$$

(2) less 3 times (1) gives:

$$9Yv^{12}=176.595$$

$$\Rightarrow Y=\frac{176.595}{9\times 0.62460}=£31.415m$$

Substituting back into (1) gives:

$$X=\frac{(81.109-31.415\times 0.62460)}{0.88900}= £69.164m$$

(iii) (a) In one year, the present value of the liabilities is:

$$10+10a_{\overline{9|}} \text{ at } 5\% \ =10+10\times 7.1078=81.078$$

Numerator of duration is $10\times 0+10(Ia)_{\overline{9|}} \ =332.347$

Duration of liabilities is therefore $\dfrac{332.347}{81.078}=4.0991$ years

Present value of assets is:

$69.164\times v^2+31.415\times v^{11}=69.164\times 0.90703+31.415\times 0.58468$
$=81.101$

Duration of assets will be:

$$\frac{2\times 69.164\times v^2+11\times 31.415\times v^{11}}{81.101}$$

$$=\frac{2\times 69.164\times 0.90703+11\times 31.415\times 0.58468}{81.101}=4.0383 \text{ years}$$

(b) One of the problems of immunisation is that there is a need to continually adjust portfolios. In this example, a change in the interest rate means that a portfolio that has a present value and duration equal to that of the liabilities at the outset does not have a present value and duration equal to that of the liabilities one year later.

The calculation was often performed well. In part (ii), many explanations were unclear and some candidates seemed confused between DMT and convexity although a correct explanation could involve either of these concepts.

8 (i) $t\le 20$:

$$v(t)=\exp\left(-\int_0^t 0.05+0.001s\,ds\right)$$

$$=\exp\left\{-\left[0.05s+\frac{0.001s^2}{2}\right]_0^t\right\}$$

$$=e^{-0.05t-0.0005t^2}$$

$t>20$:

$$v(t)=\exp\left\{-\left(\int_0^{20}\delta(s)\,ds+\int_{20}^t 0.05ds\right)\right\}$$

$$=v(20)\exp\left\{-[0.05s]_{20}^t\right\}$$

$$=e^{-1.2}e^{1-0.05t}=e^{-0.2-0.05t}$$

(ii) (a) PV = $100v(25)=100e^{-0.2-0.05\times 25}$

$$=100e^{-1.45}=£23.46$$

(b) $$100\left(1-\frac{d^{(4)}}{4}\right)^{4\times 25} = 100v(25) = 23.46$$

$$\Rightarrow d^{(4)} = 4\left(1-0.2346^{1/100}\right) = 0.05758$$

(iii) PV = $$\int_{20}^{25} 30e^{-0.015t}e^{-(0.2+0.05t)}dt$$

$$= 30e^{-0.2}\int_{20}^{25} e^{-0.065t}dt = \frac{30e^{-0.2}}{-0.065}\left[e^{-0.065t}\right]_{20}^{25}$$

$$= \frac{30e^{-0.2}}{-0.065}\left(e^{-1.625} - e^{-1.3}\right) = 28.575$$

Accumulated value = $$\frac{28.575}{v(25)} = 28.575e^{0.2+0.05\times 25} = 28.575e^{1.45} = 121.82$$

9 (i) The one-year spot rate of interest is simply 4% per annum effective.

For two-year spot rate of interest

First we need to find the price of the security, P:

$$P = 8a_{\overline{2|}} + 100v^2 \text{ at 3\% per annum effective.}$$

$$a_{\overline{2|}} = 1.91347 \quad v^2 = 0.942596$$

$$\Rightarrow P = 8\times 1.91347 + 100\times 0.942596 = 109.5673$$

Let the t-year spot rate of interest be it.

We already know that i_1 = 4%. i_2 is such that:

$$109.56736 = \frac{8}{1.04} + \frac{108}{(1+i_2)^2}$$

$$\Rightarrow (1+i_2)^{-2} = 0.943287$$

$$\Rightarrow i_2 = 0.029623 \text{ or } 2.9623\%.$$

For three-year spot rate of interest we need to find the price of the security P:

$$P = 8a_{\overline{3|}} + 100v^3 \text{ at 3\% per annum effective.}$$

$$a_{\overline{3|}} = 2.8286 \quad v^3 = 0.91514$$

$$\Rightarrow P = 8 \times 2.8286 + 100 \times 0.91514 = 114.1428$$

i_3 is such that:

$$114.1428 = \frac{8}{1.04} + \frac{8}{(1.029623)^2} + \frac{108}{(1+i_3)^3}$$

$$\Rightarrow \frac{108}{(1+i_3)^3} = 114.1428 - 15.23860 = 98.9042$$

$$\Rightarrow i_3 = 0.02976 \text{ or } 2.976\%.$$

(ii) The one year forward rate of interest beginning at the present time is clearly 4%.

The forward rate for one year beginning in one year is $f_{1,1}$ such that:

$$1.04\left(1+ f_{1,1}\right) = 1.029623^2 \Rightarrow f_{1,1} = 0.01935 = 1.935\%.$$

The forward rate for one year beginning in two years is $f_{2,1}$ such that:

$$1.029623^2\left(1+ f_{2,1}\right) = 1.02976^3 \Rightarrow f_{2,1} = 0.03003 = 3.003\%.$$

The forward rate for two years beginning in one year is $f_{1,2}$ such that:

$$1.02976^3 = 1.04\left(1+ f_{1,2}\right)^2$$

$$\Rightarrow f_{1,2} = 0.02468 = 2.468\%$$

(iii) Let the t-year "spot rate of inflation" be e_t

For each term $\frac{(1+i_t)^t}{(1+e_t)^t} = 1.02^t \Rightarrow (1+e_t)^t = \left(\frac{1+i_t}{1.02}\right)^t$

$$(1+e_1) = \frac{1.04}{1.02} \Rightarrow e_1 = 1.96\%$$

and so the value of the retail price index after one year would be 101.96

$$(1+e_2)^2 = \left(\frac{1.029623}{1.02}\right)^2 \Rightarrow e_2 = 0.943\%$$

and so the value of the retail price index after two years would be $100(1.00943)^2 = 101.90$

$$(1+e_3)^3 = \left(\frac{1.02976}{1.02}\right)^3 \Rightarrow e_3 = 0.9569\%$$

and so the value of the retail price index after three years would be $100(1.009569)^3 = 102.90$

(iv) The "spot" rates of inflation or the price index values could be used.

Clearly the expected rate of inflation in the first year is 1.96%.

The expected rate of inflation in the second year is:

$$\frac{101.90 - 101.96}{101.96} = -0.06\%.$$

The expected rate of inflation in the third year is:

$$\frac{102.90 - 101.90}{101.90} = 0.98\%$$

A common error was to assume that income only started after three years rather than "starting from the beginning of the third year".

10 (i) The price of the securities might have fallen because interest rates have risen or because their risk has increased (for example credit risk).

(ii)

Date	*Market price of securities (£)*	*X*		*Y*	
		No of securities held before purchases	*Market value of holdings before purchases (£)*	*No of securities held before purchases*	*Market value of holdings before purchases (£)*
1 April 2003	64	–	–	–	–
1 April 2004	65	100	6,500	100	6,500
1 April 2005	60	100	6,000	200	12,000
1 April 2006	65	1,100	71,500	300	19,500
1 April 2007	68	1,100	74,800	400	27,200
1 April 2008	70	1,100	77,000	500	35,000

(iii) (a) Money weighted rate of return is i where:

$$6,400(1+i)^5 + 60,000(1+i)^3 = 77,000$$

try i = 5% LHS = 77,625.70
try i = 4% LHS = 75,278.42

interpolation implies that

$$i = 0.05 - 0.01 \times \frac{77,625.70 - 77,000}{77,625.70 - 75,278.42} = 4.73\%$$

(*Note true answer is 4.736%*)

(b) Time weighted rate of return is i where using figures in above table:

$$(1+i)^5 = \frac{6,000}{6,400} \frac{77,000}{6,000 + 60,000} = 1.09375.$$

$$\Rightarrow i = 1.808\%$$

(iv) (a) Money weighted rate of return is i where:

$$6,400(1+i)^5 + 6,500(1+i)^4 + 6,000(1+i)^3 + 6,500(1+i)^2 + 6,800(1+i)$$
$$= 35,000$$

Put in i = 4.73%; LHS = 37,026.95

Therefore the money weighted rate of return for Y is less to make LHS less.

(b) Time weighted rate of return for Y uses the figures in the above table:

$$(1+i)^5 = \frac{6,500}{6,400}\frac{12,000}{6,500+6,500}\frac{19,500}{12,000+6,000}\frac{27,200}{19,500+6,500}\frac{35,000}{27,200+6,800}$$

= 1.09375.

$\Rightarrow i = 1.808\%$

(*Student may reason that the TWRRs are the same and can be derived from the security prices in which case, time would be saved.*)

(v) The money weighted rate of return was higher for X than for Y because there was a much greater amount invested when the fund was performing well than when it was performing badly.

The money weighted rate of return for X (and probably for Y) was more than the time weighted rate of return because the latter measures the rate of return that would be achieved by having one unit of money in the fund from the outset for five years: both X and Y has less in the fund in the years it performed badly.

This question was answered well but examiners were surprised by the large number of candidates who used interpolation or other trial and error methods in part (ii) when the answer had been given in the question. The examiners recommend that students pay attention to the details given in the solutions to parts (iii) and (iv). For such questions, candidates should be looking critically at the figures given/calculated and making points specific to the scenario rather than just making general statements taken from the Core Reading.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

19 April 2011 (am)

Subject CT1 — Financial Mathematics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula

$$\delta(t) = \begin{cases} 0.04 + 0.003t^2 & \text{for} \quad 0 < t \le 5 \\ 0.01 + 0.03t & \text{for} \quad 5 < t \end{cases}$$

(i) Calculate the amount to which £1,000 will have accumulated at $t = 7$ if it is invested at $t = 3$. [4]

(ii) Calculate the constant rate of discount per annum, convertible monthly, which would lead to the same accumulation as that in (i) being obtained. [3]
[Total 7]

2 A one-year forward contract on a stock is entered into on 1 January 2011 when the stock price is £68 and the risk-free force of interest is 14% per annum. The stock is expected to pay an annual dividend of £2.50 with the next dividend due in eight months' time.

On 1 April 2011, the price of the stock is £71 and the risk-free force of interest is 12% per annum. The dividend expectation is unchanged.

Calculate the value of the contract to the holder of the long forward position on 1 April 2011. [6]

3 An investment trust bought 1,000 shares at £135 each on 1 July 2005. The trust received dividends on its holding on 30 June each year that it held the shares.

The rate of dividend per share was as given in the table below:

30 June in year	*Rate of dividend per share (£)*	*Retail price index*
2005	…	121.4
2006	7.9	125.6
2007	8.4	131.8
2008	8.8	138.7
2009	9.4	145.3
2010	10.1	155.2

On 1 July 2010, the investment trust sold its entire holding of the shares at a price of £151 per share.

(i) Using the retail price index values shown in the table, calculate the real rate of return per annum effective achieved by the trust on its investment. [6]

(ii) Explain, without doing any further calculations, how your answer to (i) would alter (if at all) if the retail price index for 30 June 2008 had been greater than 138.7 (with all other index values unchanged). [2]
[Total 8]

4 The n-year spot rate of interest y_n, is given by:

$$y_n = 0.03 + \frac{n}{1000} \qquad \text{for } n = 1, 2, 3 \text{ and } 4$$

(i) Calculate the implied one-year and two-year forward rates applicable at time $t = 2$. [3]

(ii) Calculate, assuming no arbitrage:

(a) The price at time $t = 0$ per £100 nominal of a bond which pays annual coupons of 4% in arrear and is redeemed at 115% after 3 years.

(b) The 3-year par yield.

[6]
[Total 9]

5 A loan of nominal amount £100,000 was issued on 1 April 2011 bearing interest payable half-yearly in arrear at a rate of 6% per annum. The loan is to be redeemed with a capital payment of £105 per £100 nominal on any coupon date between 20 and 25 years after the date of issue, inclusive, with the date of redemption being at the option of the borrower.

An investor who is liable to income tax at 20% and capital gains tax of 35% wishes to purchase the entire loan on 1 June 2011 at a price which ensures that the investor achieves a net effective yield of at least 5% per annum.

(i) Determine whether the investor would make a capital gain if the investment is held until redemption. [3]

(ii) Explain how your answer to (i) influences the assumptions made in calculating the price the investor should pay. [2]

(iii) Calculate the maximum price the investor should pay. [5]
[Total 10]

6 The value of the assets held by a pension fund on 1 January 2010 was £10 million. On 30 April 2010, the value of the assets had fallen to £8.5 million. On 1 May 2010, the fund received a contribution payment of £7.5 million and paid out £2 million in benefits. On 31 December 2010, the value of the fund was £17.1 million.

(i) Calculate the annual effective money-weighted rate of return (MWRR) for 2010. [3]

(ii) Calculate the annual effective time-weighted rate of return (TWRR) for 2010. [3]

(iii) Explain why the MWRR is higher than the TWRR for 2010. [2]

The fund manager's bonus for 2010 is based on the return achieved by the fund over the year.

(iv) State, with reasons, which of the two rates of return calculated above would be more appropriate for this purpose. [2]
[Total 10]

7 A loan of £60,000 was granted on 1 July 1998.

The loan is repayable by an annuity payable quarterly in arrear for 20 years. The amount of the quarterly repayment increases by £100 after every four years. The repayments were calculated using a rate of interest of 8% per annum convertible quarterly.

(i) Show that the initial quarterly repayment is £1,370.41. [5]

(ii) Calculate the amount of capital repaid that was included in the payment made on 1 January 1999. [3]

(iii) Calculate the amount of capital outstanding after the quarterly repayment due on 1 July 2011 has been made. [4]
[Total 12]

8 A company has liabilities of £10 million due in three years' time and £20 million due in six years' time. The investment manager for the company is able to buy zero-coupon bonds for whatever term he requires and has adequate monies at his disposal.

(i) Explain whether it is possible for the investment manager to immunise the fund against small changes in the rate of interest by purchasing a single zero-coupon bond. [2]

The investment manager decides to purchase two zero-coupon bonds, one for a term of four years and the other for a term of 20 years. The current interest rate is 4% per annum effective.

(ii) Calculate the amount that must be invested in each bond in order that the company is immunised against small changes in the rate of interest. You should demonstrate that all three Redington conditions are met. [10]
[Total 12]

9 A company is considering investing in a project. The project requires an initial investment of three payments, each of £105,000. The first is due at the start of the project, the second six months later, and the third payment is due one year after the start of the project.

After 15 years, it is assumed that a major refurbishment of the infrastructure will be required, costing £200,000.

The project is expected to provide a continuous income stream as follows:

- £20,000 in the second year
- £23,000 in the third year
- £26,000 in the fourth year
- £29,000 in the fifth year

Thereafter the continuous income stream is expected to increase by 3% per annum (compound) at the start of each year. The income stream is expected to cease at the end of the 30th year from the start of the project.

(i) Show that the net present value of the project at a rate of interest of 8% per annum effective is £4,000 (to the nearest £1,000). [7]

(ii) Calculate the discounted payback period for the project, assuming a rate of interest of 8% per annum effective. [5]

[Total 12]

10 The annual rates of return from a particular investment, Investment A, are independently and identically distributed. Each year, the distribution of $(1+i_t)$, where i_t is the rate of interest earned in year t, is log-normal with parameters μ and σ^2.

The mean and standard deviation of i_t are 0.06 and 0.03 respectively.

(i) Calculate μ and σ^2. [5]

An insurance company has liabilities of £15m to meet in one year's time. It currently has assets of £14m. Assets can either be invested in Investment A, described above, or in Investment B which has a guaranteed return of 4% per annum effective.

(ii) Calculate, to two decimal places, the probability that the insurance company will be unable to meet its liabilities if:

(a) All assets are invested in Investment B.

(b) 75% of assets are invested in Investment A and 25% of assets are invested in Investment B. [6]

(iii) Calculate the variance of return from each of the portfolios in (ii)(a) and (ii)(b). [3]

[Total 14]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2011 examinations

Subject CT1 — Financial Mathematics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

July 2011

General comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

The general performance was slightly worse than in April 2010 but well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q3(ii) and Q6(iii) were less well answered than those that just involved calculation. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) We want $1000e^{\int_3^7 \delta(s)\, ds}$

$$=1000\, e^{\left[\int_3^5 \left(0.04+0.003s^2\right)ds+\int_5^7 (0.01+0.03s)ds\right]}$$

where $\int_3^5 \left(0.04+0.003s^2\right)ds = \left[0.04s+0.001\,s^3\right]_3^5$

$= 0.325 - 0.147 = 0.178$

and $\int_5^7 (0.01+0.03s)\,ds = \left[0.01s+\frac{0.03}{2}s^2\right]_5^7$

$=0.805-0.425=0.380$

$\Rightarrow$ accumulation at $t=7$ is

$1000e^{(0.178+0.380)}=1000e^{0.558}=1,747.17$

(ii) $1747.17\left(1-\frac{d^{(12)}}{12}\right)^{4\times 12}=1000$

$\Rightarrow d^{(12)}=0.138692$

2 Forward price of the contract is $K_0=\left(S_0-I\right)e^{\delta T}=\left(68-I\right)e^{0.14\times 1}$

where I is the present value of income during the term of the contract $=2.5e^{-0.14\times 8/12}$

$\Rightarrow K_0=\left(68-2.5e^{-0.14\times 8/12}\right)e^{0.14}=75.59919$

Forward price a new contract issued at time r (3 months) is
$K_r=\left(S_r-I^*\right)e^{\delta(T-r)}=\left(71-I^*\right)e^{0.12\times 9/12}$

(where I^* is the present value of income during the term of the contract)
$=2.5e^{-0.12\times 5/12}\Rightarrow K_{0.25}=\left(71-2.5e^{-0.05}\right)e^{0.09}=75.08435$

Value of original contract = $\left(K_r - K_0\right)e^{-\delta(T-r)}$

$$=\left(75.08435-75.59919\right)e^{-0.12\times 9/12}$$

$$=-0.47053=-47.053p$$

Many candidates failed to incorporate the change in the value of δ. Another common error was in counting the number of months.

3 (i) $$135,000=7,900\times\frac{121.4}{125.6}.\,v+8,400\times\frac{121.4}{131.8}v^2+8,800\times\frac{121.4}{138.7}v^3$$

$$+9,400\times\frac{121.4}{145.3}v^4+\left(10,100+151,000\right)\times\frac{121.4}{155.2}v^5$$

at i' % where i' = real yield

Approx yield:

$$135,000 = (7635.828+7737.178+7702.379+7853.820+126015.077)\,v^5$$

$$\Rightarrow i'\simeq 3.1\% \text{ p.a.}$$

Try $i'=3\%$, RHS $=137434.955$

Try $i'=3.5\%$, RHS $=134492.919$

$$i'=0.035-0.005\times\frac{135000-134492.919}{137434.955-134492.919}$$

$$=0.03414 \text{ (i.e. 3.4\% p.a.)}$$

(ii) The term:

$$8,800\times\frac{121.4}{RPI\ \text{(June 2008)}}$$

would have a lower value (i.e. the dividend paid on 30 June 2008 would have a lower value when expressed in June 2005 money units). The real yield would therefore be lower than 3.4% p.a.

The most common error on this question was incorrect use of the indices, e.g. many candidates inverted them. Several candidates also had difficulty in setting up the equation of *value. The examiners noted that a large number of final answers were given to excessive levels of accuracy given the approximate methods used.*

4 (i) We can find forward rates $f_{2,1}$ and $f_{2,2}$ from:

$(1+y_3)^3 = (1+y_2)^2(1+f_{2,1})$ and

$(1+y_4)^4 = (1+y_2)^2(1+f_{2,2})^2$

$\Rightarrow (1.033)^3 = (1.032)^2(1+f_{2,1})$

$\Rightarrow f_{2,1} = 3.50029\,\%$ p.a.

and $(1.034)^4 = (1.032)^2(1+f_{2,2})^2$

$\Rightarrow f_{2,2} = 3.60039\,\%$ p.a.

(ii) (a) Price per £100 nominal

$$4\left(v_{3.1\%} + v^2_{3.2\%} + v^3_{3.3\%}\right) + 115\,v^3_{3.3\%}$$

$$= 4(0.969932 + 0.938946 + 0.907192) + 115 \times 0.907192$$

$= 115.59$

(b) Let $yc_3 = 3-$year par yield

$$1 = yc_3\left(v_{3.1\%} + v^2_{3.2\%} + v^3_{3.3\%}\right) + v^3_{3.3\%}$$

$$1 = yc_3(0.969932 + 0.938946 + 0.907192) + 0.907192$$

$$\Rightarrow yc_3 = 0.032957$$

i.e. 3.2957% p.a.

5 (i) $\left(1+\frac{i^{(2)}}{2}\right)^2 = 1.05 \Rightarrow i^{(2)} = 4.939\%$ (or use tables)

$$g(1-t_1) = \frac{0.06}{1.05} \times 0.80 = 0.0457$$

So $i^{(2)} > g(1-t_1) \Rightarrow$ there is a capital gain on the contract

(ii) Since there is a capital gain, the loan is least valuable to the investor if the repayment is made by the borrower at the latest possible date. Hence, we assume redemption occurs 25 years after issue in order to calculate the minimum yield achieved.

(iii) If A is the price per £100 of loan:

$$A = 100 \times 0.06 \times 0.80\, a_{\overline{25}|}^{(2)} (1.05)^{\frac{2}{12}} + \left(105 - 0.35(105 - A)\right) v^{24\frac{10}{12}} \text{ at } 5\%$$

$$= 4.8 \times 1.012348 \times 14.0939 \times (1.05)^{\frac{2}{12}} + \left(105 - 0.35(105 - A)\right) \times 0.29771$$

Hence $A = \frac{69.0452 + 20.3187}{1 - 0.35 \times 0.29771} = 99.759$

$\Rightarrow$ Price of loan = £99,759

The majority of this question was well-answered but most candidates struggled with the two month adjustment. This adjustment needs to be directly incorporated into the equation of value. Calculating the price first without adjustment and then multiplying by $(1+i)^{1/6}$ will lead to the wrong answer.

6 (i) MWRR is given by:

$$10.0 \times (1+i) + 5.5 \times (1+i)^{8/12} = 17.1$$

Try 11%, LHS = 16.996

Try 12%, LHS $= 17.132$

$$\text{MMRR } = 0.11 + 0.01 \times \frac{17.1 - 16.996}{17.132 - 16.996} = 11.8\% \text{p.a.}$$

(ii) TWRR is given by:

$$\frac{8.5}{10.0}\times\frac{17.1}{8.5+5.5}=1+i \Rightarrow i=3.821\%\text{p.a.}$$

(iii) MWRR is higher since fund received a large (net) cash flow at a favourable time (i.e. just before the investment returns increased).

(iv) TWRR is more appropriate. Cash flows into and out of the fund are outside the control of the fund manager, and should not influence the level of bonus payable. TWRR is not distorted by amount and/or timing of cash flows whereas MWRR is.

The calculations in parts (i) and (ii) were generally well done but parts (iii) and (iv) were poorly answered (or not answered at all) even by many of the stronger candidates. In (iii) for example, candidates were expected to comment on the timing of the cashflows for this particular year.

7 (i) Let initial quarterly amount be X. Work in time units of one quarter. The effective rate of interest per time unit is

$$\frac{0.08}{4}=0.02 \text{ (i.e.2\% per quarter)}$$

So

$$60,000 = X\, a_{\overline{80|}} + 100v^{16}a_{\overline{64|}} + 100v^{32}a_{\overline{48|}} + 100v^{48}a_{\overline{32|}} + 100v^{64}a_{\overline{16|}} \text{ at } 2\%$$

$$\text{(where } a_{\overline{64|}}^{2\%} = \frac{1-v^{64}}{0.02} = 35.921415\text{)}$$

$$=39.7445X + 2,616.695465 + 1,627.606705 + 907.1436682 + 382.3097071$$

$$\Rightarrow X = \frac{60,000 - 5,533.756}{39.7445}$$

$$= £1,370.41 \text{ per quarter}$$

(ii) Interest paid at the end of the first quarter (i.e. on 1 October 1998) is

$$60,000\times 0.02 = £1,200$$

Hence, capital repaid on 1 October 1998 is

$$1370.41 - 1200 = £170.41$$

Therefore, interest paid on 1 January 1999 is

$(60000 - 170.41) \times 0.02 = 1196.59$

$\Rightarrow$ capital repaid on 1 January 1999 is

$1370.41 - 1196.59 = 173.82$

(iii) Loan outstanding at 1 July 2011 (after repayment of instalment)

$= 1670.41\, a_{\overline{12}|} + 1770.41\, v^{12}\, a_{\overline{16}|}$ at 2%

$= 1670.41 \times 10.5753 + 1770.41 \times 0.78849 \times 13.5777$

= £36,619

Candidates found this to be the most challenging question on the paper. The easiest method was to work in quarters with an effective rate of 2% per quarter. Where candidates worked using a year as the time period the most common error was to allow for an increase to payments of £100 pa when the increases were £400pa when they occurred. In part (i), the examiners were disappointed to see many attempts with incorrect and/or insufficient working end with the numerical answer that had been given in the question. A candidate who claims to have obtained a correct answer after making obvious errors in the working is not demonstrating the required level of skill and judgement and, indeed, is behaving unprofessionally.

Part (iii) was very poorly answered with surprisingly few candidates recognising the remaining loan was simply the present value of the last 28 payments.

8 (i) No, because the spread (convexity) of the liabilities would always be greater than the spread (convexity) of the assets then the 3rd Redington condition would never be satisfied.

(ii) Work in £millions

Let proceeds from four-year bond = X
Let proceeds from 20-year bond = Y

Require PV Assets = PV Liabilities

$Xv^4 + Yv^{20} = 10v^3 + 20v^6$ (1)

Require DMT Assets = DMT Liabilities

$\Rightarrow 4Xv^4 + 20Yv^{20} = 30v^3 + 120v^6$ (2)

(2) − 4 × (1)

$\Rightarrow 16Yv^{20} = 40v^6 - 10v^3$

$$\Rightarrow Y = \frac{40v^6 - 10v^3}{16v^{20}} = \frac{31.61258 - 8.88996}{7.30219} = £3.11175\text{m}$$

From (1):

$$X = \frac{10v^3 + 20v^6 - Yv^{20}}{v^4} = \frac{8.88996 + 15.80629 - 1.42016}{0.8548042} = £27.22973\text{m}$$

So amount to be invested in 4-year bond is

$Xv4$=£23.27609m

And amount to be invested in 20-year bond is

Yv^{20} = £1.42016m

Require Convexity of Assets > Convexity of Liabilities

$$\Rightarrow 20Xv^6 + 420Yv^{22} > 120v^5 + 840v^8$$

LHS = 981.869 > 712.411 = RHS

Therefore condition is satisfied and so above strategy will immunise company against small changes in interest rates.

Or state that spread of assets (t = 4 to t = 20) is greater than spread of liabilities (t = 3 to t = 6).

Part (i) was poorly answered. In part (ii) many candidates correctly derived X and Y as the proceeds from the two bonds. However, only the better candidates recognised that the amounts to be invested (as required by the question) were therefore Xv^4 and Yv^{20}.

9 (i) PV of outgo (£000s)

$$105\left(1 + v^{\frac{1}{2}} + v\right) + 200v^{15} = 366.31 \qquad \text{at } 8\%$$

PV of income

$$\bar{a}_{\overline{1}|}\left[\begin{array}{l}20v+23v^2+26v^3+29v^4\\ \qquad\qquad +29v^5 1.03\left(1+(1.03v)+(1.03v)^2+\ldots+(1.03v)^{24}\right)\end{array}\right]$$

$$=\bar{a}_{\overline{1}|}\left[20v+23v^2+26v^3+29v^4+29v^5 1.03\times\left(\frac{1-(1.03)^{25}v^{25}}{1-1.03v}\right)\right]$$

PV of income

$$=\bar{a}_{\overline{1}|}\{80.193+20.329\times 14.996\}=370.61$$

So NPV is 4.30 (=£4,300)

(ii) The NPV is very small. It is considerably less than the PV of the final year's income $\left(29\times(1.03)^{25}\times\bar{a}_{\overline{1}|}\times v^{29}=6.272\right)$; therefore the DPP must fall in the final year.

We know the DPP exists as the NPV > 0.

So DPP is $29+r$ where

$$366.31=\bar{a}_{\overline{1}|}\times\left\{80.193+20.329\times\left(\frac{1-(1.03)^{24}v^{24}}{1-1.03v}\right)\right\}$$

$$+29\times 1.03^{25}\times v^{29}\times\bar{a}_{\overline{r}|} \qquad \text{at } 8\%$$

$$\Rightarrow\quad 366.31=364.335+6.5169\bar{a}_{\overline{r}|}$$

$$\Rightarrow\quad \bar{a}_{\overline{r}|}=0.3031$$

$$\Rightarrow\quad v^r=0.97668\quad\Rightarrow\quad r=0.307$$

So the DPP is 29.31.

This question tended to separate out the stronger and weaker candidates. The most common errors in part (i) were discounting for an extra year, not including the one-year annuity factor and incorrectly calculating the geometric progression. Many candidates also lost marks through poorly presented or illegible methods that were therefore difficult for the examiners to follow. Part (ii) was poorly attempted with few candidates completing the question.

10 (i)

$$E(1+i_t) = 1.06$$

$$Var(1+i_t) = 0.03^2 = 0.0009$$

$$\Rightarrow 1.06 = e^{\left(\mu+\frac{\sigma^2}{2}\right)} \qquad (1)$$

$$0.0009 = e^{\left(2\mu+\sigma^2\right)}\left(e^{\sigma^2}-1\right) \qquad (2)$$

$$\Rightarrow \frac{(2)}{(1)^2} = \frac{0.0009}{(1.06)^2} = e^{\sigma^2}-1$$

$$\Rightarrow \sigma^2 = Ln\left(\frac{0.0009}{(1.06)^2}+1\right)$$

$$= 0.000800676 \quad (\text{and } \sigma = 0.0282962)$$

$$\Rightarrow 1.06 = e^{\left(\mu+\frac{0.000800676}{2}\right)}$$

$$\therefore \mu = Ln(1.06) - \frac{0.000800676}{2}$$

$$= 0.0578686$$

(ii) (a) Working in £m. Assets would accumulate to $14 \times 1.04 = 14.56 < 15$

$\Rightarrow$ Probability = 1.00

(b) The guaranteed portion of the fund would accumulate to

$0.25 \times 14 \times 1.04 = 3.64.$

$\therefore$ non-guaranteed portion needs to accumulate to

$$15 - 3.64 = 11.36$$

$\therefore$ we require probability that

$$(0.75 \times 14)(1 + i_t) < 11.36$$

$$= \Pr(1 + i_t) < 1.081905$$

$$= \Pr(\ln(1 + i_t) < \ln 1.081905)$$

$$= \Pr\left(\frac{\ln(1 + i_t) - 0.0578686}{0.0282962} < \frac{\ln 1.081905 - 0.0578686}{0.0282962}\right)$$

$$= \Pr(Z < 0.7370169) \text{ where } Z \sim N(0,1).$$

$$= 0.77$$

(iii) (a) Return is fixed (= 4% p.a.) $\Rightarrow$ variance of return = 0

(b) Return from portfolio = $0.25 \times 0.04 + 0.75\, i_t$

$\therefore$ Variance of return = $0.75^2 Var(i_t)$

$$= 0.75^2 \times 0.0009 = 0.00050625$$

[In monetary terms the variance of return for (iii)(b) will be $(£14m)^2 \times 0.00050625 = £^2 99,225m$ which is equivalent to a standard deviation of £315,000]

This question was generally well answered by those candidates who had left enough time to fully attempt the question. In part (i) the common errors were equating the mean to 0.06 instead of 1.06 and using 0.03 as the variance instead of 0.03^2. Part (ii) was also well answered although many candidates quoted the probability of meeting liabilities when the probability of not meeting the liabilities was asked for. Part (iii) a) was answered well by the candidates who attempted it, while part b) was not answered well. In part (iii) answers given in terms of the annual return and in terms of the monetary amounts were both fully acceptable.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

27 September 2011 (am)

Subject CT1 — Financial Mathematics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 A 91-day treasury bill is issued by the government at a simple rate of discount of 8% per annum.

Calculate the annual effective rate of return obtained by an investor who purchases the bill at issue. [3]

2 State the characteristics of index-linked government bonds. [3]

3 An individual intends to retire on his 65^{th} birthday in exactly four years' time. The government will pay a pension to the individual from age 68 of £5,000 per annum monthly in advance. The individual would like to purchase an annuity certain so that his income, including the government pension, is £8,000 per annum paid monthly in advance from age 65 until his 78^{th} birthday. He is to purchase the annuity by a series of payments made over four years quarterly in advance starting immediately.

Calculate the quarterly payments the individual has to make if the present value of these payments is equal to the present value of the annuity he wishes to purchase at a rate of interest of 5% per annum effective. Mortality should be ignored. [6]

4 A pension fund makes the following investments (£m):

1 January 2009	*1 July 2009*	*1 January 2010*
1.5	6.0	4.0

The rates of return earned on money invested in the fund were as follows:

1 January 2009 to 30 June 2009	*1 July 2009 to 31 December 2009*	*1 January 2010 to 31 December 2010*
1%	2%	5%

Assume that 1 January to 30 June and 1 July to 31 December are precise half-year periods.

(i) Calculate the time-weighted rate of return per annum effective over the two years from 1 January 2009 to 31 December 2010. [3]

(ii) Calculate the money-weighted rate of return per annum effective over the two years from 1 January 2009 to 31 December 2010. [3]
[Total 6]

5 A nine-month forward contract is issued on 1 March 2011 on a stock with a price of £9.56 per share at that date. Dividends of 20 pence per share are expected on both 1 April 2011 and 1 October 2011.

(i) Calculate the forward price, assuming a risk-free rate of interest of 3% per annum effective and no arbitrage. [4]

(ii) (a) Explain why the expected price of the share in nine months' time is not needed to calculate the forward price.

(b) Explain why the price of an option would be explicitly dependent on the variance of the share price but the price of a forward would not be. [4]

[Total 8]

6 The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is $a + bt$ where a and b are constants. An amount of £45 invested at time $t = 0$ accumulates to £55 at time $t = 5$ and £120 at time $t = 10$.

(i) Calculate the values of a and b. [5]

(ii) Calculate the constant force of interest per annum that would give rise to the same accumulation from time $t = 0$ to time $t = 10$. [2]

[Total 7]

7 An investment manager is considering investing in the ordinary shares of a particular company.

The current price of the shares is 12 pence per share. It is highly unlikely that the share will pay any dividends in the next five years. However, the investment manager expects the company to pay a dividend of 2 pence per share in exactly six years' time, 2.5 pence per share in exactly seven years' time, with annual dividends increasing thereafter by 1% per annum in perpetuity.

In five years' time, the investment manager expects to sell the shares. The sale price is expected to be equal to the present value of the expected dividends from the share at that time at a rate of interest of 8% per annum effective.

(i) Calculate the effective gross rate of return per annum the investment manager will obtain if he buys the share and then sells it at the expected price in five years' time. [6]

(ii) Calculate the net effective rate of return per annum the investment manager will obtain if he buys the share today and then sells it at the expected price in five years' time if capital gains tax is payable at 25% on any capital gains. [3]

(iii) Calculate the net effective real rate of return per annum the investment manager will obtain if he buys the share and then sells it at the expected price in five years' time if capital gains tax is payable at 25% on any capital gains and inflation is 4% per annum effective. There is no indexation allowance. [3]

[Total 12]

8 (i) State the conditions that are necessary for an insurance company to be immunised from small, uniform changes in the rate of interest. [2]

An insurance company has liabilities to pay £100m annually in arrear for the next 40 years. In order to meet these liabilities, the insurance company can invest in zero coupon bonds with terms to redemption of five years and 40 years.

(ii) (a) Calculate the present value of the liabilities at a rate of interest of 4% per annum effective.

(b) Calculate the duration of the liabilities at a rate of interest of 4% per annum effective. [5]

(iii) Calculate the nominal amount of each bond that the fund needs to hold so that the first two conditions for immunisation are met at a rate of interest of 4% per annum effective. [5]

(iv) (a) Estimate, using your calculations in (ii) (b), the revised present value of the liabilities if there were a reduction in interest rates by 1.5% per annum effective.

(b) Calculate the present value of the liabilities at a rate of interest of 2.5% per annum effective.

(c) Comment on your results to (iv) (a) and (iv) (b). [6]

[Total 18]

9 (i) Describe the information that an investor can obtain from the following yield curves for government bonds:

(a) A forward rate yield curve.
(b) A spot rate yield curve.
(c) A gross redemption yield curve. [6]

An investor is using the information from a government bond spot yield curve to calculate the present value of a corporate eurobond with a term to redemption of exactly five years. The investor will value each payment that is due from the bond at a rate of interest equal to $j = i + 0.01 + 0.001t$ where:

- t is the time in years at which the payment is due
- i is the annual t-year effective spot rate of interest from the government bond spot yield curve and $i = 0.02t$ for $t \leq 5$

The eurobond pays annual coupons of 10% of the nominal amount of the bond and is redeemed at par.

(ii) Calculate the present value of the eurobond. [6]

(iii) Calculate the gross redemption yield from the eurobond. [3]

(iv) Explain why the investor might use such a formula for j to determine the interest rates at which to value the payments from the corporate eurobond. [3]

[Total 18]

10 A country's football association is considering whether to bid to host the World Cup in 2026. Several countries aspiring to host the World Cup will be making bids. Regardless of whether the bid is successful, the association will incur various costs. For two years, starting on 1 January 2012, the association will incur costs at a rate of £2m per annum, assumed to be paid continuously, to prepare the bid.

If the football association is successful, the following costs will be incurred from 1 January 2016 until 31 December 2025:

- One stadium will be built each year for ten years. The first stadium will be built in 2016 and is expected to cost £200m; the stadium built in 2017 is expected to cost £210m; and so on, with the cost of each stadium rising by 5% each year. The costs of building each stadium are assumed to be incurred halfway through the relevant year.

- Administration costs at a rate of £100m per annum will be incurred, payable monthly in advance from 1 January 2025 until 31 December 2026.

- Revenues from television, ticket receipts, advertising and so on are expected to be £3,300m and are assumed to be received continuously throughout 2026.

(i) Explain why the payback period is not a good indicator of whether this project is worthwhile. [3]

The football association decides to judge whether to go ahead with the bid by calculating the net present value of the costs and revenues from a successful bid on 1 January 2012 at a rate of interest of 4% per annum effective.

(ii) Determine whether the association should make the bid. [13]

The football association is discussing how it might factor into its calculations the fact that it is not certain to win the right to host the World Cup because other countries are also bidding.

(iii) Explain how you might adjust the above calculations if the probability of winning the right to host the World Cup is 0.1 and whether this adjustment would make it more likely or less likely that the bid will go ahead. [3]

[Total 19]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2011 examinations

Subject CT1 — Financial Mathematics
Core Technical

Purpose of Examiners' Reports

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

December 2011

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2011 paper

The general performance was considerably better than in September 2010 and also slightly better than in April 2011. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q5(ii) and Q9(iv) were less well answered than those that just involved calculation. Marginal candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 $\left(1-\frac{91}{365}\times 0.08\right)=(1+i)^{-91/365}$

$0.980055=(1+i)^{-91/365}$

$1+i=1.08416 \Rightarrow i=8.416\%$

2 Issued by the government
Pay regular interest
Redeemable at a given redemption date
Normally liquid/marketable
More or less risk-free relative to inflation
Low expected return
Low default risk
Coupon and capital payments linked to an index of prices…
… with a time lag.

This type of bookwork question is common in CT1 exam papers. As such, it was disappointing that only about one-sixth of candidates obtained full marks here (which could be achieved by listing six distinct features).

3 Let the annual rate of payment = X

Present value of the payments = $X\ddot{a}_{\overline{4}|}^{(4)}$

Present value of the payments needed from the annuity is:

$8,000\ddot{a}_{\overline{3}|}^{(12)}v^4+3,000\ddot{a}_{\overline{10}|}^{(12)}v^7$

$X\ddot{a}_{\overline{4}|}^{(4)}=8,000\ddot{a}_{\overline{3}|}^{(12)}v^4+3,000\ddot{a}_{\overline{10}|}^{(12)}v^7$

$a_{\overline{3}|}=2.7232 \quad i/d^{(4)}=1.031059$

$a_{\overline{4}|}=3.5460 \quad a_{\overline{10}|}=7.7217 \quad i/d^{(12)}=1.026881 \quad v^4=0.82270 \quad v^7=0.71068$

$X\frac{i}{d^{(4)}}a_{\overline{4}|}=8,000\frac{i}{d^{(12)}}a_{\overline{3}|}v^4+3,000\frac{i}{d^{(12)}}a_{\overline{10}|}v^7$

$$X\times 1.031059\times 3.5460=8,000\times 1.026881\times 2.7232\times 0.82270$$
$$+3,000\times 1.026881\times 7.7217\times 0.71068$$

$3.65614X=18,404.80+16.905.51$

$X = £9,657.81$

$\therefore$ Quarterly payment is: £2,414.45.

Many candidates struggled to allow correctly for the Government pension. In some cases, candidates would have scored more marks if they had explained their methodology and their workings more clearly.

4 (i) The fund value on 30 June 2009 will be:

$$1.5 \times 1.01 = 1.515$$

The fund value on 31 December 2009 will be:

$$(1.5 \times 1.01 + 6) \times 1.02 = 7.6653$$

The fund value on 31 December 2010 will be:

$$\left[(1.5 \times 1.01 + 6) \times 1.02 + 4\right] \times 1.05 = 12.2486$$

TWRR is i such that:

$$\frac{1.515}{1.5} \times \frac{7.6653}{7.515} \times \frac{12.2486}{11.6653} = (1+i)^2 = 1.0817$$

$$\therefore i = 4.005\%$$

(This can also be calculated directly from the rates of return for which no marks would be lost).

(ii) The equation of value is:

$$1.5(1+i)^2 + 6.0(1+i)^{1\frac{1}{2}} + 4(1+i) = 12.2486$$

Try i = 4% LHS = 12.146

Try i = 4.5% LHS = 12.22754

Try i = 5% LHS = 12.3094

Interpolate:

$$i = 0.045 + \frac{12.2486 - 12.22754}{12.3094 - 12.22754} \times 0.005$$

= 0.04629 or 4.63%

A common error was to assume that the 1% and 2% rates of return were annualised figures rather than returns over a six-month period.

5 (i) Forward price is accumulated value of the share less the accumulated value of the expected dividends:

$$F = 9.56(1.03)^{9/12} - 0.2(1.03)^{8/12} - 0.2(1.03)^{2/12}$$

= 9.7743 – 0.20398 – 0.20099

= £9.3693

(ii) (a) Although the share will be bought in nine months, it is not necessary to take into account the expected share price. The current share price already makes an allowance for expected movements in the price and the investor is simply buying an instrument that is (more or less) identical to the underlying share but with deferred payment. As such, under given assumptions, the forward can be priced from the underlying share.

(b) An option does not have to be exercised. As such, movements in the share price in one direction will benefit the holder whereas movements in the other direction will not harm him. The more volatile is the underlying share price, the more potential there is for gain for the holder of the option (with limited risk of loss), compared with holding the underlying share. This is not the case for a forward which has to be exercised.

Part (i) was well-answered but part (ii) was very poorly answered. The examiners anticipated that many candidates would find part (ii)(b) challenging but it was pleasing to see some of the strongest candidates give some well-reasoned explanations for this part.

6 (i) $45e^{\int_0^5 (a+bt)dt} = 55 \quad (1)$

$45e^{\int_0^{10} (a+bt)dt} = 120 \quad (2)$

From (1)

$$45\exp\left[at + \frac{bt^2}{2}\right]_0^5 = 55$$

$$\ln\left(\frac{55}{45}\right) = 5a + 12.5b = 0.2007 \qquad (1a)$$

From (2)

$$45\exp\left[at + \frac{bt^2}{2}\right]_0^{10} = 120$$

$$\ln\left(\frac{120}{45}\right) = 10a + 50b = 0.98083$$

From (1a)

$$10a = 0.4014 - 25b \qquad (2a)$$

Substituting into (2a)

$$0.4014 + 25b = 0.98083$$

$$\therefore b = \frac{0.98083 - 0.4014}{25} = 0.02318$$

Substituting into (1a)

$$5a + 12.5 \times 0.02318 = 0.2007$$

$$\therefore a = \frac{0.2007 - 12.5 \times 0.0231772}{5} = -0.01781$$

(ii) $45e^{10\delta} = 120$

$$e^{10\delta} = \frac{120}{45};10\delta = \ln\left(\frac{120}{45}\right) = 0.98083$$

$$\therefore \delta = 0.09808 \text{ or } 9.808\ \%$$

7 (i) Expected price of the shares in five years is:

$$X = 2v + 2.5v^2 + 2.5 \times 1.01 \times v^3 + 2.5 \times 1.01^2 v^4 + \ldots$$

$$= 2v + 2.5v^2 + 2.5v^2 \left(1.01v + 1.01^2 v^2 + \ldots.\right)$$

$1.01v + 1.01^2 v^2 + \ldots$ at 8% $= \dfrac{1}{i'}$

where $i' = \dfrac{1.08}{1.01} - 1 = 0.069307$

$$X = 2 \times 0.92593 + 2.5 \times 0.85734 + \frac{2.5 \times 0.85734}{0.069307}$$

= 3.9952 + 30.9254 = 34.9206

Equation of value for the investor is:

$$12(1+i)^5 = 34.9206$$

$i = 0.23817$ or 23.817%

(ii) $12(1+i)^5 = 34.9206 - (34.9206 - 12) \times 0.25$

where i is the net rate of return.

$$12(1+i)^5 = 29.1905$$

$i = 0.1946$ or 19.46%

(iii) The cash flow received in nominal terms is still the same: 29.190495

The equation of value expressed in real terms is:

$12 = \dfrac{29.1905}{(1+f)^5} v^5$ where $f = 0.04$

$$v^5 = \frac{12 \times (1.04)^5}{29.1905} = 0.50016$$

$$\therefore v = 0.50016^{1/5} = 0.87061$$

i = 14.86%

8 (i) The present value of the assets is equal to the present value of the liabilities at the starting rate of interest.

The duration /discounted mean term/volatility of the assets is equal to that of the liabilities.

The convexity of the assets (or the spread of the timings of the asset cashflows) around the discounted mean term is greater than that of the liabilities.

(ii) (a) PV of liabilities is: £100*m* $a_{\overline{40}|}$ at 4%

$= £100\text{m} \times 19.7928$

$= £1{,}979.28\text{m}$

(b) The duration of the liabilities is:

$$\sum_{t=1}^{t=40} 100t\, v^t \Big/ \sum_{t=1}^{t=40} 100v^t \quad \text{(working in £m)}$$

$$= \frac{100\sum_{t=1}^{t=40} t\, v^t}{1{,}979.28} = \frac{100(Ia)_{\overline{40}|}}{1{,}979.28} \text{ at } 4\%$$

$$= \frac{100 \times 306.3231}{1{,}979.28} = 15.4765 \text{ years}$$

(iii) Let x = nominal amount of five-year bond
y = nominal amount of 40-year bond.

working in £m

$1{,}979.28 = xv^5 + yv^{40}$ — (1)

$30{,}632.31 = 5xv^5 + 40yv^{40}$ — (2)

multiply equation (1) by 5.

$9{,}896.4 = 5xv^5 + 5yv^{40}$ — (1a)

subtract (1a) from (2) to give

$20735.91 = 35yv^{40}$

$$\frac{20{,}735.91}{35 \times v^{40}} = y$$

with $v^{40} = 0.20829$

$y = 2,844.38$

Substitute into (1) to give:

$1,979.28 = Xv^5 + 2,844.38 \times 0.20829$

$v^5 = 0.82193$

$$\frac{1,979.28 - 2,844.38 \times 0.20829}{0.82193} = x = 1,687.28$$

Therefore £1,687.28m nominal of the five-year bond and £2,844.38*m* nominal of the 40-year bond should be purchased.

(iv) (a) The duration of the liabilities is 15.4765

Therefore the volatility of the liabilities is $\frac{15.4765}{1.04}$ =14.88125%

The value of the liabilities would therefore change by:

$1.5 \times 0.1488125 \times 1,979.28m = £441.81m$

and the revised present value of the liabilities will be £2,421.09*m*.

(b) PV of liabilities is: £100*m* $a_{\overline{40|}}$ at 2.5%

$= £100\text{m} \times \frac{1 - 1.025^{-40}}{0.025}$

= £2,510.28*m*.

(c) The PV of liabilities has increased by £531*m*. This is significantly greater than that estimated in (iv) (a). This estimation will be less valid for large changes in interest rates as in this case.

The first three parts were generally well-answered but, in part (iv), the examiners were surprised that so few candidates were able to use the duration to estimate the change in the value of the liability.

9 (i) (a) The theoretical rate of return that could be achieved over a given time period in the future from investment in government bonds today.

(b) The theoretical rate of return that could be achieved between the current time and a given future time from investment in government bonds.

(c) The gross redemption yield that could be theoretically achieved by investing in government bonds of different terms to redemption. The yield curve represents a statistical average gross redemption yield.

(ii)

Time	*Government bond yield*	*Valuation rate of interest*	*P.V factor*
1	0.02	0.031	0.96993
2	0.04	0.052	0.90358
3	0.06	0.073	0.80947
4	0.08	0.094	0.69812
5	0.1	0.115	0.58026

$$PV = 10\left(0.96993 + 0.90358 + 0.80947 + 0.69812 + 0.58026\right) + 100 \times 0.58026$$

= 97.6396.

(iii) GRY is such that: $97.6396 = 10a_{\overline{5|}} + 100v^5$

Try 11% $a_{\overline{5|}} = 3.69590$ $v^5 = 0.59345$ RHS = 96.30397

Try 10% $a_{\overline{5|}} = 3.7908$ $v^5 = 0.62092$ RHS = 100 [calculation not necessary]

Interpolate to find i:

$$i = -\frac{97.6396 - 96.30397}{100 - 96.30397} \times 0.01 + 0.11$$

$\Rightarrow i = 0.10639$ or 10.64%

(iv) It is reasonable for the investor to price a corporate bond with reference to the rates of return from government bonds which may be (more or less) risk free.

A risk premium will then need to be added.

It is also not unreasonable that this risk premium rises with term as the uncertainty regarding credit risk rises.

This question proved to be the most difficult on the paper. The examiners had anticipated that some candidates would have difficulty with part (i) but it was disappointing to see the number of candidates who were unable to give even a basic description of a spot rate and a forward rate. Part (iv) was also very poorly answered and whilst it had been anticipated that only the *strongest candidates would make all the relevant points, the examiners were surprised at how many candidates failed to score any marks on this part.*

10 (i) The payback period measures the earliest time at which the project breaks even but takes no account either of interest on borrowings or on cash flows received after the payback period. It is therefore a poor measure of ultimate profitability.

(ii) The present value of preparation costs is (in £m):

$2\bar{a}_{\overline{2}|} @ 4\%$ per annum effective.

$= 2.\frac{i}{\delta}.a_{\overline{2}|} \qquad \frac{i}{\delta} = 1.019869 \;\; a_{\overline{2}|} = 1.8861$

$= 2 \times 1.019869 \times 1.8861 = 3.847$

The present value the stadium building costs is (in £m):

$$200v^{4\frac{1}{2}} + 200 \times 1.05v^{5\frac{1}{2}} + 200 \times 1.05^2 v^{6\frac{1}{2}} + \ldots + 200 \times 1.05^9 v^{13\frac{1}{2}}$$

$$200v^{4\frac{1}{2}}\left(1 + 1.05v + 1.05^2 v^2 + \ldots + 1.05^9 v^9\right)$$

$$= 200v^{4\frac{1}{2}}\left[\frac{1 - 1.05^{10} v^{10}}{1 - 1.05v}\right]$$

with $v = 0.96154 \quad v^{10} = 0.67556 \quad 1.05^{10} = 1.62889 \quad v^{4\frac{1}{2}} = 0.83820$

$$= 200 \times 0.83820 \times \left(\frac{1 - 1.62889 \times 0.67556}{1 - 1.05 \times 0.96154}\right)$$

$= £1,750.837$

Present value of admin. costs is (£m):

$100\ddot{a}^{(12)}_{\overline{2}|}v^{13} \qquad @ 4\%$

with $\frac{i}{d^{(12)}} = 1.021537 \; v^{13} = 0.60057 \; a_{\overline{2}|} = 1.8861$

$= 100 \times 1.021537 \times 1.8861 \times 0.60057$

$= 115.714$

Present value of revenue (£m):

$3,300\bar{a}_{\overline{1|}}v^{14}$ with $\dfrac{i}{\delta}=1.019869$ $a_{\overline{1|}}=0.9615$ $v^{14}=0.57748$

$=3,300\times 1.019869\times 0.9615\times 0.57748$

= 1,868.781

NPV = 1,868.781 – 115.714 – 1,750.837 – 3.847 = –£1.617m.

Therefore should not make a bid.

(iii) One way of dealing with this would be to multiply the NPV of all the revenues and costs that are only received if the bid is won by 0.1.

The costs of preparing the bid would be incurred for certain and therefore not multiplied by 0.1. This adjustment would make it less likely the bid will go ahead because the only certain item is a cost.

This question contained a potential ambiguity regarding the timing of the administration costs. Although the examiners felt that the approach given in the model solution was the most logical, candidates who assumed that the administration costs were only payable during 2025 were given full credit. This question was answered well and it was very pleasing to see that (a) candidates managed their time efficiently and so left enough time to make a good attempt at the question with the most marks and (b) candidates who made calculation errors still clearly explained their method and so were able to pick up significant marks for their working.

END OF EXAMINER'S REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

24 April 2012 (am)

Subject CT1 – Financial Mathematics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 In a particular bond market, n-year spot rates can be approximated by the function $0.06 - 0.02e^{-0.1n}$.

(i) Calculate the gross redemption yield for a 3-year bond which pays coupons of 3% annually in arrear, and is redeemed at par. Show all workings. [6]

(ii) Calculate the 4-year par yield. [3]
[Total 9]

2 The value of the assets held by an investment fund on 1 January 2011 was £2.3 million.

On 30 April 2011, the value of the assets had risen to £2.9 million and, on 1 May 2011, there was a net cash inflow to the fund of £1.5 million. On 31 December 2011, the value of the assets was £4.2 million.

(i) Calculate the annual effective time-weighted rate of return (TWRR) for 2011. [2]

(ii) Calculate, to the nearer 0.1%, the annual effective money-weighted rate of return (MWRR) for 2011. [4]

(iii) Explain why the TWRR is significantly higher than the MWRR for 2011. [2]
[Total 8]

3 A company has borrowed £500,000 from a bank. The loan is to be repaid by level instalments, payable annually in arrear for ten years from the date the loan is made. The annual instalments are calculated at an effective rate of interest of 9% per annum.

(i) Calculate:

(a) the amount of the level annual instalments.

(b) the total amount of interest which will be paid over the ten-year term. [3]

At the beginning of the eighth year, immediately after the seventh instalment has been made, the company asks for the loan to be rescheduled over a further four years from that date. The bank agrees to do this on condition that the rate of interest is increased to an effective rate of 12% per annum for the term of the rescheduled instalments and that repayments are made quarterly in arrear.

(ii) (a) Calculate the amount of the new quarterly instalment.

(b) Calculate the interest content of the second quarterly instalment of the rescheduled loan repayments.
[5]
[Total 8]

4 (i) Explain what is meant by the "no arbitrage" assumption in financial mathematics. [2]

An investor entered into a long forward contract for a security four years ago and the contract is due to mature in five years' time. The price of the security was £7.20 four years ago and is now £10.45. The risk-free rate of interest can be assumed to be 2.5% per annum effective throughout the nine-year period.

(ii) Calculate, assuming no arbitrage, the value of the contract now if the security will pay dividends of £1.20 annually in arrear until maturity of the contract. [3]

(iii) Calculate, assuming no arbitrage, the value of the contract now if the security has paid and will continue to pay annually in arrear a dividend equal to 3% of the market price of the security at the time of payment. [3]
[Total 8]

5 An investor is considering two projects, Project A and Project B. Project A involves the investment of £1,309,500 in a retail outlet. Rent is received quarterly in arrear for 25 years, at an initial rate of £100,000 per annum. It is assumed that the rent will increase at a rate of 5% per annum compound, but with increases taking place every five years. Maintenance and other expenses are incurred quarterly in arrear, at a rate of £12,000 per annum. The retail outlet reverts to its original owner after 25 years for no payment.

Project B involves the purchase of an office building for £1,000,000. The rent is to be received quarterly in advance at an initial rate of £85,000 per annum. It is assumed that the rent will increase to £90,000 per annum after 20 years. There are no maintenance or other expenses. After 25 years the property reverts to its original owner for no payment.

(i) Show that the internal rate of return for project A is 9% per annum effective. [5]

(ii) Calculate the annual effective internal rate of return for Project B. Show your working. [4]

(iii) Discuss the extent to which the answers to parts (i) and (ii) above will influence the investor's decision over which project to choose. [3]
[Total 12]

6 A fixed-interest bond pays annual coupons of 5% per annum in arrear on 1 March each year and is redeemed at par on 1 March 2025.

On 1 March 2007, immediately after the payment of the coupon then due, the gross redemption yield was 3.158% per annum effective.

(i) Calculate the price of the bond per £100 nominal on 1 March 2007. [3]

On 1 March 2012, immediately after the payment of the coupon then due, the gross redemption yield on the bond was 5% per annum.

(ii) State the new price of the bond per £100 nominal on 1 March 2012. [1]

A tax-free investor purchased the bond on 1 March 2007, immediately after payment of the coupon then due, and sold the bond on 1 March 2012, immediately after payment of the coupon then due.

(iii) Calculate the gross annual rate of return achieved by the investor over this period. [2]

(iv) Explain, without doing any further calculations, how your answer to part (iii) would change if the bond were due to be redeemed on 1 March 2035 (rather than 1 March 2025). You may assume that the gross redemption yield at both the date of purchase and the date of sale remains the same as in parts (i) and (ii) above. [3]

[Total 9]

7 The annual yields from a fund are independent and identically distributed. Each year, the distribution of $1 + i$ is log-normal with parameters $\mu = 0.05$ and $\sigma^2 = 0.004$, where i denotes the annual yield on the fund.

(i) Calculate the expected accumulation in 20 years' time of an annual investment in the fund of £5,000 at the beginning of each of the next 20 years. [5]

(ii) Calculate the probability that the accumulation of a single investment of £1 made now will be greater than its expected value in 20 years' time. [5]

[Total 10]

8 The force of interest, $\delta(t)$, at time t is given by:

$$\delta(t) = \begin{cases} 0.04 + 0.003t^2 & \text{for} \quad 0 < t \leq 5 \\ 0.01 + 0.03t & \text{for} \quad 5 < t \leq 8 \\ 0.02 & \text{for} \quad t > 8 \end{cases}$$

(i) Calculate the present value (at time t = 0) of an investment of £1,000 due at time t = 10. [4]

(ii) Calculate the constant rate of discount per annum convertible quarterly, which would lead to the same present value as that in part (i) being obtained. [2]

(iii) Calculate the present value (at time t = 0) of a continuous payment stream payable at the rate of $100e^{0.01t}$ from time t = 10 to t = 18. [4]

[Total 10]

9 An ordinary share pays dividends on each 31 December. A dividend of 35p per share was paid on 31 December 2011. The dividend growth is expected to be 3% in 2012, and a further 5% in 2013. Thereafter, dividends are expected to grow at 6% per annum compound in perpetuity.

(i) Calculate the present value of the dividend stream described above at a rate of interest of 8% per annum effective for an investor holding 100 shares on 1 January 2012. [4]

An investor buys 100 shares for £17.20 each on 1 January 2012. He expects to sell the shares for £18 on 1 January 2015.

(ii) Calculate the investor's expected real rate of return.

You should assume that dividends grow as expected and use the following values of the inflation index:

Year:	2012	2013	2014	2015
Inflation index at start of year:	110.0	112.3	113.2	113.8

[5]

[Total 9]

10 A company has the following liabilities:

- annuity payments of £200,000 per annum to be paid annually in arrear for the next 20 years
- a lump sum of £300,000 to be paid in 15 years

The company wishes to invest in two fixed-interest securities in order to immunise its liabilities.

Security A has a coupon rate of 9% per annum and a term to redemption of 12 years. Security B has a coupon rate of 4% per annum and a term to redemption of 30 years.

Both securities are redeemable at par and pay coupons annually in arrear. The rate of interest is 8% per annum effective.

(i) Calculate the present value of the liabilities. [3]

(ii) Calculate the discounted mean term of the liabilities. [4]

(iii) Calculate the nominal amount of each security that should be purchased so that Redington's first two conditions for immunisation against small changes in the rate of interest are satisfied for this company. [8]

(iv) Describe the further calculations that will be necessary to determine whether the company is immunised against small changes in the rate of interest. [2]
[Total 17]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2012 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

July 2012

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2012 paper

The general performance was broadly similar to the previous two exams. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q2(iii), Q5(iii) and Q6(iv) were less well answered than those that just involved calculation. Marginal candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) Price, P, of £100 nominal stock is:

$$P = 3v_{y_1} + 3v_{y_2}^2 + 103v_{y_3}^3$$

where

$y_1 = 0.041903$

$y_2 = 0.043625$

$y_3 = 0.045184$

$\Rightarrow P = 95.845$

And gross redemption yield, i%, solves:

$95.845 = 3a_{\overline{3}|} + 100v^3$ at i%

Try 4% RHS = 97.225
5% RHS = 94.554

$$\Rightarrow i = 0.04 + 0.01 \times \frac{97.225 - 95.845}{97.225 - 94.554}$$

$= 0.0452$

i.e. 4.5% p.a.

(ii) y_1, y_2 and y_3 as above. $y_4 = 0.046594$

$$1 = (yc_4)\left(v_{y_1} + v_{y_2}^2 + v_{y_3}^3 + v_{Y_4}^4\right) + v_{y_4}^4$$

$\Rightarrow 1 = yc_4 \times 3.587225 + 0.8334644$

$\Rightarrow yc_4 = 0.04642$ i.e. 4.642% p.a.

2 (i) TWRR, i, is given by:

$$\frac{2.9}{2.3} \times \frac{4.2}{2.9 + 1.5} = 1 + i \Rightarrow i = 0.204 \text{ or } 20.4\% \text{ p.a.}$$

(ii) MWRR, i, is given by:

$$2.3 \times (1+i) + 1.5(1+i)^{8/12} = 4.2$$

Then, we have:

$$\left.\begin{array}{l} i=12\% \quad \Rightarrow \quad LHS=4.1937 \\ i=13\% \quad \Rightarrow \quad LHS=4.2263 \end{array}\right\} \Rightarrow i=0.12+(0.13-0.12)\times\left(\frac{4.2-4.1937}{4.2263-4.1937}\right)$$

$= 0.122$

or 12.2% p.a.

(iii) The MWRR is lower as fund performs better before the cash inflow than after. Then, as the fund is larger after the cash inflow on 1 May 2011, the effect of the poor investment performance after this date is more significant in the calculation of the MWRR.

The calculations were performed well but the quality of the explanations in part (iii) was often poor. This type of explanation is commonly asked for in CT1 exams. To get full marks, candidates should address the specific situation given in the question rather than just repeat the bookwork.

3 (i) Let R = annual repayment

$$500,000 = R\, a_{\overline{10}|}^{9\%} = R\times 6.4177$$

$$\Rightarrow R = 77,910.04$$

and total interest $=10\times 77,910.04-500,000$

$$=279,100$$

(ii) (a) Capital outstanding at beginning of 8th year is:

$$77910.04\, a_{\overline{3}|}^{9\%} = 77909.53\times 2.5313$$

$$=197,213.28$$

Let R' be new payment per annum then

$$R'\, a_{\overline{4}|}^{(4)} = R'\times 1.043938\times 3.0373 = 197,213.28$$

$$\Rightarrow \quad R' = 62,196.62$$

and quarterly payment is £15,549.16

(b) Interest content of 2nd quarterly payment is:

$$15549.16 \times \left(1 - v_{12\%}^{3\frac{3}{4}}\right) = 5383.41$$

[Or Capital in 1st quarterly payment is

$$15549.16 - 197213.28 \times \left(1.12^{\frac{1}{4}} - 1\right) = 9,881.77$$

So capital outstanding after 1st quarterly payment

$$= 197213.28 - 9881.77 = 187331.51$$

$\Rightarrow$ Interest in next payment is

$$187331.51 \times \left(1.12^{\frac{1}{4}} - 1\right) = 5383.41]$$

Generally answered well but a number of candidates made errors in calculating the remaining term in part (ii)

4 (i) The "no arbitrage" assumption means that neither of the following applies:

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss;

nor

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

(ii) The current value of the forward price of the old contract is:

$$7.20 \times (1.025)^4 - 1.20\, a_{\overline{5}|}^{2\frac{1}{2}\%}$$

whereas the current value of the forward price of a new contract is:

$$10.45 - 1.20\, a_{\overline{5}|}^{2\frac{1}{2}\%}$$

Hence, current value of old forward contract is:

$$10.45 - 7.20 \times (1.025)^4 = £2.5025$$

(iii) The current value of the forward price of the old contract is:

$$7.20(1.025)^4(1.03)^{-9} = 6.0911$$

whereas the current value of the forward price of a new contract is:

$$10.45(1.03)^{-5} = 9.0143$$

$\Rightarrow$ current value of old forward contract is:

$$9.0143 - 6.0911 = £2.9232$$

This was the most poorly answered question on the paper but well-prepared candidates still scored full marks. Some candidates in part (ii) assumed that the dividend income was received during the lifetime of the forward contract. Whilst the examiners did not believe that such an approach was justified, candidates who assumed this alternative treatment of the income were not penalised. It was very clear that the poor performance on the question was not as result of this alternative interpretation.

5 (i) The equation of value is:

$$1309.5 = 100\left(a_{\overline{5}|}^{(4)} + (1.05)^5 v^5 a_{\overline{5}|}^{(4)} + \ldots + (1.05)^{20} v^{20} a_{\overline{5}|}^{(4)}\right) - 12a_{\overline{25}|}^{(4)}$$

Rearranging:

$$1309.5 = 100a_{\overline{5}|}^{(4)}\left(\frac{1-(1.05v)^{25}}{1-(1.05v)^5}\right) - 12a_{\overline{25}|}^{(4)}$$

At 9%, RHS is:

$$100 \times 1.033144 \times 3.8897 \times \left(\frac{1-\left(\frac{1.05}{1.09}\right)^{25}}{1-\left(\frac{1.05}{1.09}\right)^5}\right) - 12 \times 1.033144 \times 9.8226$$

$$= 401.8570 \times \frac{0.607292}{0.170505} - 121.7779$$

$= 1309.53 \Rightarrow$ IRR is 9% p.a.

(ii) For Project B the equation of value is

$$1000 = 85\ddot{a}_{\overline{20}|}^{(4)} + v^{20}\ 90\ddot{a}_{\overline{5}|}^{(4)}$$

Roughly $1000 \simeq 85a_{\overline{25}|} \Rightarrow i \simeq 7\%$

At 7% RHS is 1039.05
$= 85 \times 1.043380 \times 10.5940 + 0.25842 \times 1.043380 \times 4.1002 \times 90$

8% RHS is 956.78
$= 85 \times 1.049519 \times 9.8181 + 0.21455 \times 1.049519 \times 3.9927 \times 90$

$\Rightarrow i \simeq 7.5\%$ p.a.

(iii) Project A is more attractive since it has the higher IRR. However, the investor will also need to take into account other factors such as:

- the outlay is much higher for Project A than Project B
- the interest rate at which the investor might need to borrow at to finance a project since it will affect the net present values and discounted payback periods of the projects
- the risks for each project that the rents and expenses will not be those assumed in the calculations.

In part (i) candidates were asked to demonstrate that the internal rate of return was a given value. In such questions, candidates should set up the equation of value and clearly show each stage of their algebra and their calculations (including the evaluation of all factors that make up the equation). Many candidates claimed that they had shown the correct answer despite obvious errors and/or insufficient working. Candidates who tried to create a "proof" where the arguments didn't follow logically gained few marks. In this type of question, if you can't complete a proof, it is better to show how far you have got and be open about being unable to proceed further. This will generally gain more intermediate markst.

Part (ii) was answered well but in part (iii) few candidates came up with any of the other factors that should be considered.

6 (i) Price per £100 nominal is given by:

$$P = 5 \times a_{\overline{18}|}^{3.158\%} + 100v_{3.158\%}^{18} = 5 \times \left(\frac{1 - v_{3.158\%}^{18}}{0.03158} \right) + 100v_{3.158\%}^{18} = 125.00$$

(ii) As coupons are payable annually and the gross redemption yield is equal to the annual coupon rate, the new price per £100 nominal is £100.

i.e. $P = 5a_{\overline{13}|}^{5\%} + 100v_{5\%}^{13} = 5\left(\frac{1 - v_{5\%}^{13}}{0.05} \right) + 100v_{5\%}^{13} = 100.00$

(iii) Equation of value is:

$$125.00 = 5a_{\overline{5}|} + 100v^5 \Rightarrow i = 0\%$$

Thus, the investor makes a return of 0% per annum over the period.

(iv) Longer-dated bonds are more volatile.

Thus, as a result of the rise in gross redemption yields from 3.158% per annum on 1 March 2007 to 5% on 1 March 2012, the fall in the price of the bond would be greater.

Thus, as the income received over the period would be unchanged, the overall return achieved would be reduced (as a result of the greater fall in the capital value).
[In fact, the price on 1 March 2007 would have been £133.91 per £100 nominal falling to £100 per £100 nominal on 1 March 2012.

i.e. in this case, we need to find *i* such that $133.91 = 5a_{\overline{5}|} + 100V^5 \Rightarrow i < 0\%$.]

The first three parts were generally well-answered although relatively few candidates noticed that parts (ii) and (iii) could be answered quickly and consequently many candidates made avoidable calculation errors.

7 (i) $E(1+i) = e^{\mu + \frac{1}{2}\sigma^2}$

$= e^{0.05 + \frac{1}{2} \times 0.004}$

$= 1.0533757$

$\therefore E[i] = 0.0533757$ since $E(1+i) = 1 + E(i)$

Let A be the accumulation of £5000 at the end of 20 years

then $E[A] = 5000\,\ddot{s}_{\overline{20|}}$ at rate $j = 0.0533757$

$$= 5000\frac{\left((1+j)^{20}-1\right)}{j}\times(1+j)$$

$$= 5000\frac{\left(1.0533757^{20}-1\right)}{0.0533757}\times 1.0533757$$

$$= £180{,}499$$

(ii) Let the accumulation be S_{20}

S_{20} has a log-normal distribution with parameters 20μ and $20\sigma^2$

$$\therefore E[S_{20}] = e^{20\mu+\frac{1}{2}\left(20\sigma^2\right)} \qquad \left\{\text{or}\,(1+j)^{20}\right\}$$

$$= \exp(20\times 0.05 + 10\times 0.004)$$

$$= e^{1.04} \quad = 2.829217$$

$$\ln S_{20} \sim N\left(20\mu, 20\sigma^2\right)$$

i.e. $\ln S_{20} \sim N(1,\ 0.08)$

$$P\left(S_{20} > e^{1.04}\right) = P(\ln S_{20} > 1.04)$$

$$= P\left(Z > \frac{1.04-1}{\sqrt{0.08}}\right) \quad \text{where} \quad Z \sim N(0,1)$$

$$= P(Z > 0.14) = 1 - \Phi(0.14)$$

$$= 1 - 0.56 = 0.44$$

Questions regarding annual investments are comparatively rarely asked on this topic and students seemed to struggle with part (i). Part (ii) was answered better in general than equivalent questions in previous exams.

8 (i) for $t > 8$

$$v(t) = \exp -\left\{\int_0^5 0.04 + 0.003t^2 dt + \int_5^8 0.01 + 0.03t dt + \int_8^t 0.02 dt\right\}$$

$$= \exp -\left\{\left[0.04t + 0.001t^3\right]_0^5 + \left[0.01t + 0.015t^2\right]_5^8 + \left[0.02t\right]_8^t\right\}$$

$$= \exp -\left\{0.2 + 0.125 + 0.01\times 3 + 0.015\left(8^2 - 5^2\right) + 0.02t - 0.02\times 8\right\}$$

$$= \exp -\left\{0.325 + 0.615 + 0.02t - 0.16\right\}$$

$$= e^{-(0.78+0.02t)}$$

Hence PV of £1,000 due at t = 10 is:

$$1000\times \exp -\left(0.78 + 0.02\times 10\right) = £375.31$$

(ii) $$1000\left(1-\frac{d^{(4)}}{4}\right)^{4\times 10} = 375.31$$

$$\left(1-\frac{d^{(4)}}{4}\right)^{40} = \frac{375.31}{1000}$$

$$d^{(4)} = 4\left(1-\left(\frac{375.31}{1000}\right)^{1/40}\right)$$

$$\underline{= 0.09681}$$

(iii) $$PV = \int_{10}^{18} \rho(t)v(t)\,dt$$

$$= \int_{10}^{18} 100e^{0.01t} \times e^{-(0.78+0.02t)}$$

$$= 100e^{-0.78}\int_{10}^{18} e^{-0.01t} dt$$

$$= 100e^{-0.78}\left\{\left[\frac{e^{-0.01t}}{-0.01}\right]_{10}^{18}\right\}$$

$$= \frac{100}{0.01} e^{-0.78}\left(e^{-0.1} - e^{-0.18}\right)$$

$$= £318.90$$

Parts (i) and (ii) were answered well. Some candidates made errors in part (iii) by not discounting the payment stream back to time 0.

9

(i)

$$PV = 100 \times 0.35\left(1.03v + 1.03 \times 1.05v^2 + 1.03 \times 1.05 \times 1.06v^3 + 1.03 \times 1.05 \times 1.06^2 v^4 + \cdots\right)$$

$$= 35\left(1.03v + 1.03 \times 1.05v^2 + \frac{1.03 \times 1.05 \times 1.06v^3}{1 - 1.06v}\right) \quad @ \ 8\%$$

$$= 35\left(\frac{1.03}{1.08} + \frac{1.03 \times 1.05}{1.08^2} + \frac{1.03 \times 1.05 \times 1.06}{1.08^3} \times \frac{1.08}{0.02}\right)$$

$$= 35\left(0.95370 + 0.92721 + 49.14223\right)$$

$$= £1785.81$$

(ii) Real rate of return is i such that:

$$1720 = 35\left(1.03 \times \frac{110}{112.3}v + 1.03 \times 1.05 \times \frac{110}{113.2}v^2 + 1.03 \times 1.05 \times 1.06 \times \frac{110}{113.8}v^3\right) + 1800 \times \frac{110}{113.8}v^3$$

$$= 35\left(1.0089047v + 1.050928v^2 + 1.108110v^3\right) + 1739.894552v^3$$

$$= 35.3116645v + 36.78248v^2 + 1778.678402v^3$$

For initial estimate, assume all income received at end of 3 years:

1720 ≈ 1850.77v3

$\Rightarrow v \approx 0.9758696 \quad \Rightarrow i \approx 2.4727$

Try i = 2.5%, RHS = 1721.14 ≈ 1720

so i = 2.5%

Most candidates made a good attempt at part (i) although slight errors in setting up the equation and/or in the calculation were common. Many candidates struggled with setting up the required equation in part (ii).

10 (i) Working in 000's

$$PV_L = 200a_{\overline{20|}} + 300v^{15} \quad @ \ 8\%$$
$$= 200 \times 9.818147 + 300 \times 0.315242$$
$$= 2058.20199$$

i.e. £2,058,201.99

(ii)

$$DMT_L = \frac{200v + 200 \times 2v^2 + 200 \times 3v^3 + \cdots + 200 \times 20v^{20} + 300 \times 15v^{15}}{200a_{\overline{20|}} + 300v^{15}}$$
$$= \frac{200(Ia)_{\overline{20|}} + 300 \times 15v^{15}}{2058.20199} \quad @ \ 8\%$$
$$= \frac{200 \times 78.9079 + 300 \times 15 \times 0.31524}{2058.20199}$$
$$= \frac{17200.175}{2058.20199} = 8.3569 \text{ years}$$

(iii) Redington's first two conditions are:

$$\Rightarrow PV_L = PV_A$$
$$\Rightarrow DMT_L = DMT_A$$

Let the nominal amount in securities A and B be X and Y respectively.

$$PV_A = PV_L \Rightarrow X\left(0.09a_{\overline{12|}} + v^{12}\right) + Y\left(0.04a_{\overline{30|}} + v^{30}\right) = 2058201.99 \ @ \ 8\%$$
$$\Rightarrow X\left(0.09 \times 7.5361 + 0.39711\right) + Y\left(0.04 \times 11.2578 + 0.09938\right)$$
$$\Rightarrow 1.075361X + 0.549689Y = 2058201.99$$
$$\Rightarrow X = \frac{2058201.99 - 0.549689Y}{1.075361}$$

$$DMT_A = DMT_L \Rightarrow \frac{X\left(0.09(Ia)_{\overline{12|}} + 12v^{12}\right) + Y\left(0.04(Ia)_{\overline{30|}} + 30v^{30}\right)}{2058201.99} = 8.3569$$
$$\Rightarrow X\left(0.09(Ia)_{\overline{12|}} + 12v^{12}\right) + Y\left(0.04(Ia)_{\overline{30|}} + 30v^{30}\right) = 17200175 \quad @ \ 8\%$$
$$\Rightarrow X\left(0.09 \times 42.17 + 12 \times 0.39711\right) + Y\left(0.04 \times 114.7136 + 30 \times 0.09938\right) = 17200175$$
$$\Rightarrow 8.56066X + 7.56986Y = 17200175$$
$$\Rightarrow 8.56066 \times \frac{(2058201.99 - 0.549689Y)}{1.075361} + 7.56986Y = 17200175$$
$$\Rightarrow 3.19394Y = 815370.9$$
$$\Rightarrow Y = 255287, \ X = 1783470$$

Hence company should purchase £1,783,470 nominal of security A and £255,287 nominal of security B for Redington's first two conditions to be satisfied.

(iv) Redington's third condition is that the convexity of the asset cash flow series is greater than the convexity of the liability cash flow series. Therefore the convexities of the asset cash flows and the liability cash flows will need to be calculated and compared.

Generally well answered but candidates' workings in part (iii) were often unclear which made it difficult for examiners to award marks when calculation errors had been made.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

3 October 2012 (am)

Subject CT1 – Financial Mathematics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 An investor is considering two investments. One is a 91-day deposit which pays a rate of interest of 4% per annum effective. The second is a treasury bill.

Calculate the annual simple rate of discount from the treasury bill if both investments are to provide the same effective rate of return. [3]

2 The nominal rate of discount per annum convertible quarterly is 8%.

(i) Calculate the equivalent force of interest. [1]

(ii) Calculate the equivalent effective rate of interest per annum. [1]

(iii) Calculate the equivalent nominal rate of discount per annum convertible monthly. [2]

[Total 4]

3 An investment fund is valued at £120m on 1 January 2010 and at £140m on 1 January 2011. Immediately after the valuation on 1 January 2011, £200m is paid into the fund. On 1 July 2012, the value of the fund is £600m.

(i) Calculate the annual effective time-weighted rate of return over the two-and-a half year period. [3]

(ii) Explain why the money-weighted rate of return would be higher than the time-weighted rate of return. [2]

[Total 5]

4 A ten-month forward contract was issued on 1 September 2012 for a share with a price of £10 at that date. Dividends of £1 per share are expected on 1 December 2012, 1 March 2013 and 1 June 2013.

(i) Calculate the forward price assuming a risk-free rate of interest of 8% per annum convertible half-yearly and no arbitrage. [4]

(ii) Explain why it is not necessary to use the expected price of the share at the time the forward matures in the calculation of the forward price. [2]

[Total 6]

5 (i) State the characteristics of a Eurobond [4]

(ii) (a) State the characteristics of a certificate of deposit.

(b) Two certificates of deposit issued by a given bank are being traded. A one-month certificate of deposit provides a rate of return of 12 per cent per annum convertible monthly. A two-month certificate of deposit provides a rate of return of 24 per cent per annum convertible monthly.

Calculate the forward rate of interest per annum convertible monthly in the second month, assuming no arbitrage. [4]

[Total 8]

6 A loan is to be repaid by an increasing annuity. The first repayment will be £200 and the repayments will increase by £100 per annum. Repayments will be made annually in arrear for ten years. The repayments are calculated using a rate of interest of 6% per annum effective.

(i) Calculate the amount of the loan [2]

(ii) (a) Calculate the interest component of the seventh repayment.
(b) Calculate the capital component of the seventh repayment.

[4]

(iii) Immediately after the seventh repayment, the borrower asks to have the original term of the loan extended to fifteen years and wishes to repay the outstanding loan using level annual repayments. The lender agrees but changes the interest rate at the time of the alteration to 8% per annum effective.

Calculate the revised annual repayment. [3]

[Total 9]

7 An individual wishes to make an investment that will pay out £200,000 in twenty years' time. The interest rate he will earn on the invested funds in the first ten years will be either 4% per annum with probability of 0.3 or 6% per annum with probability 0.7. The interest rate he will earn on the invested funds in the second ten years will also be either 4% per annum with probability of 0.3 or 6% per annum with probability 0.7. However, the interest rate in the second ten year period will be independent of that in the first ten year period.

(i) Calculate the amount the individual should invest if he calculates the investment using the expected annual interest rate in each ten year period. [2]

(ii) Calculate the expected value of the investment in excess of £200,000 if the amount calculated in part (i) is invested. [3]

(iii) Calculate the range of the accumulated amount of the investment assuming the amount calculated in part (i) is invested. [2]

[Total 7]

8 The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula

$$\delta(t) = \begin{cases} 0.03 + 0.01t & \text{for} \quad 0 \le t \le 9 \\ 0.06 & \text{for} \quad 9 < t \end{cases}$$

(i) Derive, and simplify as far as possible, expressions for $v(t)$ where $v(t)$ is the present value of a unit sum of money due at time t. [5]

(ii) (a) Calculate the present value of £5,000 due at the end of 15 years.

(b) Calculate the constant force of interest implied by the transaction in part (a). [4]

A continuous payment stream is received at rate $100e^{-0.02t}$ units per annum between $t = 11$ and $t = 15$.

(iii) Calculate the present value of the payment stream. [4]

[Total 13]

9 (i) Describe three theories that have been put forward to explain the shape of the yield curve. [7]

The government of a particular country has just issued five bonds with terms to redemption of one, two, three, four and five years respectively. The bonds are redeemed at par and have coupon rates of 4% per annum payable annually in arrear.

(ii) Calculate the duration of the one-year, three-year and five-year bonds at a gross redemption yield of 5% per annum effective. [6]

(iii) Explain why a five-year bond with a coupon rate of 8% per annum would have a lower duration than a five-year bond with a coupon rate of 4% per annum. [2]

Four years after issue, immediately after the coupon payment then due the government is anticipating problems servicing its remaining debt. The government offers two options to the holders of the bond with an original term of five years:

Option 1: the bond is repaid at 79% of its nominal value at the scheduled time with no final coupon payment being paid.

Option 2: the redemption of the bond is deferred for seven years from the original redemption date and the coupon rate reduced to 1% per annum for the remainder of the existing term and the whole of the extended term.

Assume the bonds were issued at a price of £95 per £100 nominal.

(iv) Calculate the effective rate of return per annum from Options 1 and 2 over the total life of the bond and determine which would provide the higher rate of return. [6]

(v) Suggest two other considerations that bond holders may wish to take into account when deciding which options to accept. [2]

[Total 23]

10 Two investment projects are being considered.

(i) Explain why comparing the two discounted payback periods or comparing the two payback periods are not generally appropriate ways to choose between two investment projects. [3]

The two projects each involve an initial investment of £3m. The incoming cash flows from the two projects are as follows:

Project A

In the first year, Project A generates cash flows of £0.5m. In the second year it will generate cash flows of £0.55m. The cash flows generated by the project will continue to increase by 10% per annum until the end of the sixth year and will then cease. Assume that all cash flows are received in the middle of the year.

Project B

Project B generates cash flows of £0.64m per annum for six years. Assume that all cash flows are received continuously throughout the year.

(ii) (a) Calculate the payback period from Project B.

(b) Calculate the discounted payback period from Project B at a rate of interest of 4% per annum effective.

[5]

(iii) Show that there is at least one “cross-over point” for Projects A and B between 0% per annum effective and 4% per annum effective where the cross-over point is defined as the rate of interest at which the net present value of the two projects is equal. [6]

(iv) Calculate the duration of the incoming cash flows from Projects A and B at a rate of interest of 4% per annum effective. [6]

(v) Explain why the net present value of Project A appears to fall more rapidly than the net present value of Project B as the rate of interest increases. [2]

[Total 22]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2012 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

D C Bowie
Chairman of the Board of Examiners

December 2012

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2012 paper

The general performance was of a lower standard compared with the previous two exams. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q3(ii), Q4(ii) and Q9(iii) were less well answered than those that just involved calculation. This is an area to which attention should be paid. Candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations. At the other end of the spectrum, there was a difficulty for many candidates when it came to answering questions involving introductory ideas.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 Let d be the annual simple rate of discount.

The discounted value of 100 in the deposit account would be x such that:

$$x = 100(1.04)^{-91/365} = 99.0269$$

$\therefore$ to provide the same effective rate of return a treasury bill that pays 100 must have a price of 99.0269 and $100\left(1-\frac{91}{365}\times d\right) = 99.0269$

$$\therefore\ 1-\frac{91}{365}\times d = \frac{99.0269}{100} = 0.990269$$

$$d = (1\text{-}0.990269) \times \frac{365}{91} = 0.03903$$

Many candidates scored full marks on this question but many others failed to score any marks at all. Some candidates incorrectly used (1-*nd*) *as an accumulation factor*

2 (i) $e^{-\delta/4} = 1-\frac{0.08}{4} = 0.98 \quad \therefore \delta = 0.080811$

(ii) $(1+i)^{-1} = \left(1-\frac{0.08}{4}\right)^4 = 0.92237 \therefore\ i = 0.084166$

(iii) $\left(1-\frac{d^{(12)}}{12}\right)^{12} = \left(1-\frac{0.08}{4}\right)^4 = 0.92237 \therefore d^{(12)} = 0.080539$

A lot of marginal candidates scored very badly on this question even though it was covering an introductory part of the syllabus.

3 (i) $(1+i)^{2.5} = \frac{140}{120}\times\frac{600}{140+200} = 2.05882$

$\therefore$ 1 + i = 1.33490

$\therefore$ i = 33.49% p.a. effective.

(ii) The money weighted rate of return weights performance according to the amount of money in the fund. The fund was performing better after it had been given the large injection of money on 1/1/2011.

Part (i) was answered well. The type of explanation asked for in part (ii) is commonly asked for in CT1 exams. To get full marks, candidates should address the specific situation given in the question rather than just repeat the bookwork.

4 (i) Present value of dividends, I, is:

$$I = \left(v^{1/4} + v^{1/2} + v^{3/4}\right)$$

Calculated at $i'\%$ when $(1+i') = (1.04)^2 = 1.0816$

So $I = 1.0816^{-1/4} + 1.0816^{-1/2} + 1.0816^{-3/4}$
$= 2.88499$

Hence, forward price, F, is:

$$F = (10-2.88499)(1+i')^{10/12} \text{ at } 8.16\%$$
$$= (10-2.88499) \times 1.0816^{10/12} = £7.5956$$

(ii) The price of the forward can be determined from the price of the share (for which it is a close substitute). The forward is like the share but with delayed settlement and without dividends.

5 (i) The characteristics of a Eurobond are:

- Medium- or long-term borrowing
- Unsecured
- Regular coupon payments
- Redeemed at par
- Issued and traded internationally/not in the jurisdiction of any one country
- Can be denominated in any currency (e.g. not the currency of issuer)
- Tend to be issued by large companies, governments or supra-national organisations
- Yields depend on issue size and issuer (or marketability and risk)
- Issue characteristics may vary – market free to allow innovation

(ii) (a) The characteristics of a certificate of deposit are:

- Tradable certificate issued by banks stating that money has been deposited
- Terms to maturity between one and six months
- Interest payable on maturity/issued at a discount
- Security and marketability will depend on issuing bank
- Active secondary market

(b) Answer is i such that $(1.01)^{12}\left(1+\dfrac{i^{(12)}}{12}\right)^{12}=(1.02)^{24}$ giving

$i^{(12)} = 36.119\%$

6 (i) Amount of loan is:

$100(Ia)_{\overline{10|}} + 100a_{\overline{10|}}$ at 6% p.a.

= 100 ×36.9624 + 100 × 7.3601
= 3696.24 + 736.01 = £4,432.25

(ii) (a) the o/s loan after sixth instalment is:

$100(Ia)_{\overline{4|}} + 700\,a_{\overline{4|}}$

=100 × 8.4106 + 700 × 3.4651 = 841.06 + 2425.57 = £3,266.64

The interest component is therefore:

0.06 × 3266.64 = £196.00

(b) The capital component =

800-196.00 = £604.00

(iii) The capital remaining after the seventh instalment is
3266.64 – 604.00 = 2662.64

Let the new instalment = X

$Xa_{\overline{8|}} = 2{,}662.64$ at 8%

$a_{\overline{8|}} = 5.7466$; X = 2,662.64/5.7466 = £463.34

7 (i) Expected annual interest rate in both ten-year periods =
0.04 ×0.3 + 0.06 × 0.7 = 0.054 or 5.4%

Amount of the investment would be X such that:

$X\,(1.054)^{20} = 200{,}000$

$\therefore X$ = £69,858.26

(ii) Expected accumulation factors in both ten-year periods are:

$0.3\,(1.04)^{10} + 0.7(1.06)^{10} = 1.697667$

The accumulation factors in each ten-year period are independent.

Therefore the expected accumulation is:

$69{,}858.26 \times 1.697667 \times 1.697667$

$= £201{,}336.55$

Therefore the value of investment over and above £200,000 = £1,336.55.

(iii) The extreme outcomes for the investment are:

$69{,}858.26 \times 1.04^{20} = 153{,}068.06$
$69{,}858.26 \times 1.06^{20} = 224{,}044.91.$

Therefore the range is: £70,976.85

Many candidates struggled with this question and seemed to have difficulty particularly with part (ii). Part (iii) was also badly answered even though part (ii) was not needed to answer part (iii).

8 (i) $t \le 9$

$$v(t) = e^{-\int_0^t (0.03+0.01s)ds}$$

$$= e^{-\left[0.03s+\frac{0.01s^2}{2}\right]_0^t}$$

$$= e^{-\left[0.03t+0.005t^2\right]}$$

$t > 9$

$$V(t) = e^{-\left[\int_0^9 \delta(s)ds+\int_9^t 0.06ds\right]}$$

$$= V(9).e^{0.06(t-9)}$$

$$= e^{-0.675}.e^{-0.06(t-9)}$$

$$= e^{-(0.135+0.06t)}$$

(ii) (a) $PV = 5,000\, e^{-(0.135+0.06\times 15)}$

$$= 5,000\, e^{-1.035}$$
$$= £1,776.13$$

(b) $1,776.13\, e^{\delta\times 15} = 5,000$

$$e^{\delta\times 15} = 2.81511$$

$$15\delta = \ln 2.81511$$

$$\delta = \frac{\ln 2.81511}{15} = 0.0690$$

(iii)

$$P.V. = \int_{11}^{15} e^{-(0.135+0.06t)} \times 100\, e^{-0.02t}\, dt$$
$$= \int_{11}^{15} 100\, e^{-0.135-0.08t}\, dt$$
$$= 100\, e^{-0.135} \left[\frac{e^{-0.08t}}{-0.08} \right]_{11}^{15}$$
$$= 100\, e^{-0.135} (5.18479 - 3.76493)$$
$$= 124.055$$

Generally answered well but some candidates lost marks in part (i) by not deriving the discount factor for t < 9.

9 (i) Expectations theory: yields on short and long-term bonds are determined by expectations of future interest rates as it is assumed that a long-term bond is a substitute for a series of short-term bonds.
[If interest rates are expected to rise (fall) long-term bonds will have higher (lower) yields that short-term bonds.]

Liquidity preference: it is assumed that investors have an inherent preference for short-term bonds because interest-rate sensitivity is lower. As such, (there is an upward bias on the expectations-based yield curve) and longer-term bonds will offer a higher expected return than implied by expectations theory on its own. *N.B. the part in brackets is not in core reading.*

Market segmentation: bonds of different terms to redemption are attractive to different investors with different liabilities.

The supply of bonds of different terms to redemption will depend on the strategy of the relevant issuer. The term structure is determined by the interaction of supply and demand in each term-to-redemption segment.

(ii) Duration = $\dfrac{\sum tC_t\, v^t}{\sum C_t\, v^t} = \dfrac{4\times(Ia)_{\overline{n}|} + 100n\,v^n}{4\times a_{\overline{n}|} + 100\,v^n}$

For n = 1 to 5. Clearly duration on one-year bond is one year.

Term	$(Ia)_{\overline{n}\|}$	$100v^n$	$a_{\overline{n}\|}$	$n100v^n$	$4(Ia)_{\overline{n}\|}$
3	5.3580	86.384	2.7232	259.152	21.432
5	12.5664	78.353	4.3295	391.765	50.2656

Duration of three-year bond:

$$\frac{21.432+259.152}{4\times 2.7232+86.384} = 2.884 \text{ years}$$

Duration of five-year bond:

$$\frac{50.2656+391.765}{4\times 4.3295+78.353} = 4.620 \text{ years}$$

(iii) The duration of a bond is the average time of the cashflows weighted by present value. The coupon payments of the 8% coupon bond will be a higher proportion of the total proceeds than for the 4% coupon bond. Thus, a greater proportion of the total proceeds of the 8% coupon bond will be received before the end of the term. The average time of the cashflows will be shorter and hence the duration will be lower.

(iv) **Option 1**

The equation of value would be:

$$95 = 4a_{\overline{4}|} + 79v^5$$

The rate of return is zero (incoming and outgoing cash flows are equal).

Option 2

The equation of value would be:

$$95 = 4a_{\overline{4}|} + v^4\, a_{\overline{8}|} + 100v^{12}$$

$i = 2.5\%$

RHS = 4 × 3.762 + 0.90595 × 7.1701 + 100 × 0.74356
= 15.0479 + 6.4958 + 74.3556 = 95.8993

i = 3%

RHS = 4 × 3.7171 + 0.88849 × 7.0197 + 100 ×0.70138
= 14.8684 + 6.2369 + 70.1380
= 91.2433

By interpolation:

$$i = 0.005 \times \left(\frac{95.8998 - 95}{95.8998 - 91.2433} \right) + 0.025$$

= 0.025966 or 2.6% per annum effective.

Hence Option 2 would provide the higher rate of return

(v) Two of the following:

- Option 2 creates a higher duration bond which might not be suitable for the investor
  ….e.g. alternative investments may be available in the longer term
- The credit risk over the longer duration may be greater
- The inflation risk over the longer duration may be greater
- There may be tax implications because of the differing capital and income combinations.
- the institution could reinvest the proceeds from option 1 at whatever rate of return prevails.

Part (i) was often poorly answered even though this was bookwork and candidates also struggled with part (ii). In part (ii) it is important to include the correct units for the duration (in this case, years). Most candidates made a good attempt at part (iv) even if some made calculation errors (e.g. in the calculation of the outstanding term of the bond under Option 2). Marginal candidates scored badly on parts (iii) and (v).

10 (i) The payback period simply looks at the time when the total incoming cash flows are greater than the total outgoing cash flows. It takes no account of interest at all.

Though the discounted payback period takes account of interest that would have to be paid on loans, it only looks at when loans used to finance outgoing cash flows would be repaid and not at the overall profitability of the projects.

(ii) (a) Outgoing cash flow = £3m

In £m, at time t, total incoming cash flows are £0.64t

We need t such that $3 = 0.64t$

$$t = \tfrac{3}{0.64} = 4.6875 \text{ years}$$

(b) Present value of incoming cash flows at time t is:

$$0.64\bar{a}_{\overline{t}|} = 0.64\left(\frac{1-v^t}{\delta}\right) \text{where } \delta = 0.039221$$

Require t such that:

$$0.64\left(\frac{1-v^t}{0.039221}\right) = 3$$

$$1 - v^t = 0.183848$$
$$v^t = 0.816152$$
$$t \ln v = \ln 0.816152$$

$$t = \frac{\ln 0.816152}{\ln v}$$

$$= -\frac{-0.203155}{-0.039221} = 5.1798 \text{ years}$$

(iii) Crossover point is the rate of interest at which the n.p.v. of the two projects is equal. As the present value of the cash outflows for both projects is the same at all rates of interest, the crossover point is the rate of interest at which the present value of the cash inflows from both projects is equal.

P.V of cash inflows from Project B = $0.64\bar{a}_{\overline{6}|}$

P.V of cash inflows from Project A =

$$0.5\, v^{\frac{1}{2}} + 1.1 \times 0.5 \times v^{1\frac{1}{2}} + \cdots + 1.1^5 \times 0.5 \times v^{5\frac{1}{2}}$$
$$= 0.5v^{\frac{1}{2}}\left[\frac{1-1.1^6 \times v^6}{1-1.1 \times v}\right]$$

Therefore require i such that:

$$0.64\, \bar{a}_{\overline{6}|} - 0.5\, v^{\frac{1}{2}}\left[\frac{1-1.1^6 v^6}{1-1.1v}\right] = 0$$

Let $i = 4\%$

$$a_{\overline{6}|} = 5.2421 \quad i/\delta = 1.019869$$

$$v^{1/2} = 0.98058 \quad v = 0.96154$$
$$v^6 = 0.79031$$
$$1.1^6 = 1.77156$$

$$\text{LHS} = 0.64 \times 5.2421 \times 1.019869 - 0.5 \times 0.98058 \left[\frac{1 - 1.77156 \times 0.79031}{1 - 1.1 \times 0.96154}\right]$$
$$= 3.4216 - 0.49029 \times 6.93490 = 3.4216 - 3.4001$$
$$= 0.0215$$

Let $i = 0\%$

$$\text{LHS} = 0.64 \times 6 - 0.5 \times \left[\frac{1 - 1.77156}{1 - 1.1}\right] = 3.84 - 3.8578 = -0.0178$$

Given that NPV of Project A is greater than that of project B at 0% per annum effective and the reverse is true at 4% per annum effective, the NPV of the two projects must be equal at some point between 0% and 4%.

(iv) **Project A**

Duration is:

$$\frac{v^{1/2} 0.5(0.5 + 1.1 \times 1.5v + 1.1^2 \times 2.5v^2 + 1.1^3 \times 3.5v^3 + 1.1^4 \times 4.5v^4 + 1.1^5 \times 5.5 \times v^5)}{0.49029 \times 6.93490}$$

Term in brackets is

0.5 + 1.58654 + 2.79678 + 4.14139 + 5.63183 + 7.28047 = 21.93702.

$$\therefore \text{ Duration} = \frac{0.98059 \times 0.5 \times 21.93702}{0.49029 \times 6.93490} = 3.163 \text{ years}$$

Project B

$$\text{Duration is}: \frac{0.64 \int_0^6 t v^t \, dt}{0.64 \int_0^6 v^t \, dt} = \frac{(\bar{I}\bar{a})_{\overline{6}|}}{\bar{a}_{\overline{6}|}} = \frac{\left(i/\delta \; a_{\overline{6}|} - 6v^6\right)/\delta}{i/\delta \; a_{\overline{6}|}}$$

$$= \frac{(1.019869 \times 5.2421 - 6 \times 0.79031)/0.039221}{1.019869 \times 5.2421}$$

$$\frac{15.41000}{5.3462} = 2.882 \text{ years}$$

(v) Project A has a longer duration and therefore the present value of its incoming cash flows is more sensitive to changes in the rate of interest. As such, when the interest rate rises, the present value of incoming cash flows falls more rapidly than for Project B.

Most candidates could calculate the discounted payback period but struggled with the undiscounted equivalent. As in Q9, the units should be included within the answer. The working of many candidates in part (iii) was often unclear even when the formulae were correctly derived. In part (iv) many candidates incorrectly thought the duration should be $\frac{(I\bar{a})_{\overline{6}|}}{\bar{a}_{\overline{6}|}}$ *for Project B.*

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

15 April 2013 (pm)

Subject CT1 – Financial Mathematics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 The value of the assets held by an investment fund on 1 January 2012 was £1.3 million.

On 30 September 2012, the value of the assets was £1.9 million.
On 1 October 2012, there was a net cash outflow from the fund of £0.9 million.
On 31 December 2012, the value of the assets was £0.8 million.

(i) Calculate the annual effective time-weighted rate of return (TWRR) for 2012. [2]

(ii) Calculate the annual effective money-weighted rate of return (MWRR) for 2012 to the nearest 1%. [3]

(iii) Explain why the MWRR is significantly higher than the TWRR. [2]
[Total 7]

2 (i) Explain the main difference:

(a) between options and futures.
(b) between call options and put options.
[4]

(ii) A one-year forward contract is issued on 1 April 2013 on a share with a price at that date of £10.50. Dividends of £1.10 per share are expected on 30 September 2013 and 31 March 2014. On 1 April 2013, the 6-month risk-free spot rate of interest is 4.5% per annum convertible half-yearly and the 12-month risk-free spot rate of interest is 5% per annum convertible half-yearly.

Calculate the forward price at issue, stating any further assumptions made. [4]
[Total 8]

3 Three bonds each paying annual coupons in arrear of 6% and redeemable at £103 per £100 nominal reach their redemption dates in exactly one, two and three years' time, respectively. The price of each bond is £97 per £100 nominal.

(i) Calculate the gross redemption yield of the 3-year bond. [3]

(ii) Calculate the one-year and two-year spot rates implied by the information given. [3]
[Total 6]

4 An investor is interested in purchasing shares in a particular company.

The company pays annual dividends, and a dividend payment of 30 pence per share has just been made.

Future dividends are expected to grow at the rate of 5% per annum compound.

(i) Calculate the maximum price per share that the investor should pay to give an effective return of 9% per annum. [4]

(ii) Without doing any further calculations, explain whether the maximum price paid will be higher, lower or the same if:

(a) after consulting the managers of the company, the investor increases his estimate of the rate of growth of future dividends to 6% per annum.

(b) as a result of a government announcement, the general level of future price inflation in the economy is now expected to be 2% per annum higher than previously assumed.

(c) general economic uncertainty means that, whilst the investor still estimates future dividends will grow at 5% per annum, he is now much less sure about the accuracy of this assumption.

You should consider the effect of each change separately. [6]
[Total 10]

5 The force of interest per unit time at time t, $\delta(t)$, is given by:

$$\delta(t) = \begin{cases} 0.1 - 0.005t & \text{for } t < 6 \\ 0.07 & \text{for } t \geq 6 \end{cases}$$

(i) Calculate the total accumulation at time 10 of an investment of £100 made at time 0 and a further investment of £50 made at time 7. [4]

(ii) Calculate the present value at time 0 of a continuous payment stream at the rate £$50e^{0.05t}$ per unit time received between time 12 and time 15. [5]
[Total 9]

6 A cash sum of £10,000 is invested in a fund and held for 15 years. The yield on the investment in any year will be 5% with probability 0.2, 7% with probability 0.6 and 9% with probability 0.2, and is independent of the yield in any other year.

(i) Calculate the mean accumulation at the end of 15 years. [2]

(ii) Calculate the standard deviation of the accumulation at the end of 15 years. [5]

(iii) Without carrying out any further calculations, explain how your answers to parts (i) and (ii) would change (if at all) if:

(a) the yields had been 6%, 7% and 8% instead of 5%, 7%, and 9% per annum, respectively.

(b) the investment had been held for 13 years instead of 15 years.

[4]
[Total 11]

7 An insurance company has liabilities of £6 million due in 8 years' time and £11 million due in 15 years' time. The assets consist of two zero-coupon bonds, one paying £X in 5 years' time and the other paying £Y in 20 years' time. The current interest rate is 8% per annum effective. The insurance company wishes to ensure that it is immunised against small changes in the rate of interest.

(i) Determine the values of £X and £Y such that the first two conditions for Redington's immunisation are satisfied. [8]

(ii) Demonstrate that the third condition for Redington's immunisation is also satisfied. [2]
[Total 10]

8 A car manufacturer is to develop a new model to be produced from 1 January 2016 for six years until 31 December 2021. The development costs will be £19 million on 1 January 2014, £9 million on 1 July 2014 and £5 million on 1 January 2015.

It is assumed that 6,000 cars will be produced each year from 2016 onwards and that all will be sold.

The production cost per car will be £9,500 during 2016 and will increase by 4% each year with the first increase occurring in 2017. All production costs are assumed to be incurred at the beginning of each calendar year.

The sale price of each car will be £12,600 during 2016 and will also increase by 4% each year with the first increase occurring in 2017. All revenue from sales is assumed to be received at the end of each calendar year.

(i) Calculate the discounted payback period at an effective rate of interest of 9% per annum. [9]

(ii) Without doing any further calculations, explain whether the discounted payback period would be greater than, equal to, or less than the period calculated in part (i) if the effective rate of interest were substantially less than 9% per annum. [2]

[Total 11]

9 A fixed-interest security pays coupons of 8% per annum half yearly on 1 January and 1 July. The security will be redeemed at par on any 1 January from 1 January 2017 to 1 January 2022 inclusive, at the option of the borrower.

An investor purchased a holding of the security on 1 May 2011, at a price which gave him a net yield of at least 6% per annum effective. The investor pays tax at 30% on interest income and 25% on capital gains.

On 1 April 2013 the investor sold the holding to a fund which pays no tax at a price to give the fund a gross yield of at least 7% per annum effective.

(i) Calculate the price per £100 nominal at which the investor bought the security. [5]

(ii) Calculate the price per £100 nominal at which the investor sold the security. [3]

(iii) Show that the effective net yield that the investor obtained on the investment was between 8% and 9% per annum. [6]

[Total 14]

10 A loan is repayable by annual instalments in arrear for 20 years. The initial instalment is £5,000, with each subsequent instalment decreasing by £200.

The effective rate of interest over the period of the loan is 4% per annum.

(i) Calculate the amount of the original loan. [3]

(ii) Calculate the capital repayment in the 12th instalment. [3]

After the 12th instalment is paid, the borrower and lender agree to a restructuring of the debt.

The £200 reduction per year will no longer continue. Instead, future instalments will remain at the level of the 12th instalment and the remaining term of the debt will be shortened. The final payment will then be a reduced amount which will clear the debt.

(iii) (a) Calculate the remaining term of the revised loan.
(b) Calculate the amount of the final reduced payment.
(c) Calculate the total interest paid during the term of the loan.

[8]
[Total 14]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2013 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

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D C Bowie
Chairman of the Board of Examiners

July 2013

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2013 paper

This paper proved to be marginally more challenging than other recent papers and the general performance was of a slightly lower standard compared with the previous April exams. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q1(iii) and Q4(ii) were less well answered than those that just involved calculation. This is an area to which attention should be paid. Candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) TWRR, i, is given by:

$$\frac{1.9}{1.3} \times \frac{0.8}{1.9-0.9} = 1+i \Rightarrow i = 0.169 \text{ or } 16.9\% \text{ p.a.}$$

(ii) MWRR, i, is given by:

$$1.3 \times (1+i) - 0.9 \times (1+i)^{\frac{3}{12}} = 0.8$$

Then, we have:

$$\left.\begin{array}{lll} i = 30\% & \Rightarrow & LHS = 0.729 \\ i = 40\% & \Rightarrow & LHS = 0.841 \end{array}\right\} \Rightarrow i \approx 0.3 + (0.4-0.3) \times \left(\frac{0.8-0.729}{0.841-0.729}\right) = 0.36$$

or 36% p.a.

(iii) MWRR is higher as fund performs much better before the cash outflow than after. As the fund is smaller after 1 October 2012, the effect of the poor investment performance is less significant.

The calculations were performed well but the quality of the explanations in part (iii) was often poor. A common error was to cite the large withdrawal itself as the reason for the superior MWRR.

2 (i) (a) Options – holder has the right but not the obligation to trade.

Futures – both parties have agreed to the trade and are obliged to do so.

(b) Call Option – right but not the obligation to BUY specified asset in the future at specified price.

Put Option – right but not the obligation to SELL specified asset in the future at specified price.

(ii) Assume no arbitrage.

The present value of the dividends, I, is:

$$I = 1.1v_{2.25\%} + 1.1v_{2.5\%}^2 = 1.1 \times (0.977995 + 0.951814)$$
$$= 2.12279$$

Hence, forward price $F = (10.50 - 2.12279) \times 1.025^2$
$= £8.8013$

3 (i) $97 = 6\,a_{\overline{3}|} + 103v^3$

Try 9% RHS = 94.723
Try 8% RHS = 97.227

Interpolation gives

$$0.08 + \frac{97.227 - 97}{97.227 - 94.723} \times 0.01$$

$= 0.08091$

i.e. 8.09% p.a. (exact answer is 8.089%)

(ii) Let i_n = spot rate for term n

Then $97 = 109v_{i_1\%}$

$\Rightarrow i_1 = 12.371\%$ p.a.

$$97 = 6v_{i_1\%} + 109v_{i_2\%}^2$$

$$109(1+i_2)^{-2} = 97 - \frac{6}{1.12371}$$

$\Rightarrow i_2 = 9.049\%$ p.a.

Part (i) was generally well answered. Some candidates wasted time in (ii) through using linear interpolation to solve the yield for the one year bond.

4 (i) Maximum price payable by investor is given by:

$$P = 0.30 \times 1.05 \times v_{9\%} + 0.30 \times 1.05^2 \times v_{9\%}^2 + \ldots$$

$$= 0.30 \times \left(\frac{1.05}{1.09}\right) \times \left[1 + \left(\frac{1.05}{1.09}\right) + \left(\frac{1.05}{1.09}\right)^2 + \ldots\right]$$

$$= 0.30 \times \left(\frac{1.05}{1.09}\right) \times \frac{1}{1 - \frac{1.05}{1.09}}$$

$$= 0.30 \times \left(\frac{1.05}{1.09}\right) \times \frac{1.09}{0.09 - 0.05} = 0.30 \times \frac{1.05}{0.09 - 0.05}$$

$$= £7.875$$

(ii) (a) Increasing the expected rate of dividend growth, g , will increase the maximum price that the investor is prepared to pay to purchase the share since the dividend income is expected to be higher.

(b) An increase in the expected rate of future price inflation is likely to lead to an increase in both the expected rate of dividend growth (as nominal level of profits should increase in line with inflation) and the nominal return required from the investment (as the investor is likely to want to maintain the required real return).

Thus, the maximum price that the investor is prepared to pay will be (largely) unchanged – in fact, it will increase slightly due to (1 + g) term in numerator.

(c) If the investor is more uncertain about the rate of future dividend growth (whilst the expected dividend growth is unchanged), then the required return, i, is likely to be increased to compensate for the increased uncertainty.

Thus, the maximum price that the investor is prepared to pay will reduce.

Part (i) was generally well answered although common errors included adding an extra 30 pence dividend at the start or to assume that the first dividend was payable immediately.

The examiners expected candidates to find part (ii) challenging and this was indeed the case with very few candidates scoring full marks. In (ii)(b) full marks were awarded for a reasoned argument that led to a final answer of either an increase or no change in the price. In general, some credit was given for valid reasoning even if the final conclusion was incorrect.

5 (i) Accumulated value at time 10 is:

$$100 \times \exp\left(\int_0^{10} \delta(t)\,dt\right) + 50 \times \exp\left(\int_7^{10} \delta(t)\,dt\right)$$

$$= 100 \times \exp\left(\int_0^{6} (0.1 - 0.005t)\,dt + \int_6^{10} 0.07dt\right) + 50 \times \exp\left(\int_7^{10} 0.07dt\right)$$

$$= 100 \times \exp\left(\left[0.1t - 0.0025t^2\right]_{t=0}^{t=6} + \left[0.07t\right]_{t=6}^{t=10}\right) + 50 \times \exp\left(\left[0.07t\right]_{t=7}^{t=10}\right)$$

$$= 100\times\exp\left([0.6-0.09]+0.28\right)+50\times\exp(0.21)$$

$$= 220.34+61.68$$

$$= £282.02$$

(ii) Present value at time 0 is:

$$= \int_{12}^{15}\rho(t)v(t)\,dt$$

$$= \int_{12}^{15}50e^{0.05t}\times\exp\left(-\int_0^t\delta(s)\,ds\right)dt$$

$$= \int_{12}^{15}50e^{0.05t}\times\exp\left(-\left[\int_0^6(0.1-0.005s)\,ds+\int_6^t 0.07ds\right]\right)dt$$

$$= \int_{12}^{15}50e^{0.05t}\times\exp\left(-\left[\left[0.1s-0.0025s^2\right]_{s=0}^{s=6}+\left[0.07s\right]_{s=6}^{s=t}\right]\right)dt$$

$$= \int_{12}^{15}50e^{0.05t}\times\exp\left(-\left[0.51+(0.07t-0.42)\right]\right)dt$$

$$= \int_{12}^{15}50e^{0.05t}\times e^{-0.09-0.07t}dt$$

$$= 50e^{-0.09}\times\int_{12}^{15}e^{-0.02t}dt$$

$$= 50e^{-0.09}\times\left[\frac{e^{-0.02t}}{-0.02}\right]_{t=12}^{t=15}$$

$$= 2{,}500e^{-0.09}\times\left(e^{-0.24}-e^{-0.30}\right)$$

$$= £104.67$$

6 (i) $j = 0.05 \times 0.2 + 0.07 \times 0.6 + 0.09 \times 0.2$

$= 0.07$

$\Rightarrow$ mean accumulation $= 10{,}000 \times (1 + j)^{15}$

$= 10{,}000 \times (1.07)^{15}$

$= £27{,}590.32$

(ii) $s^2 = 0.05^2 \times 0.2 + 0.07^2 \times 0.6 + 0.09^2 \times 0.2 - 0.07^2$

$= 0.00506 - 0.00490$

$= 0.00016$

Var (accumulation) $= 10{,}000^2\{(1 + 2j + j^2 + s^2)^{15} - (1 + j)^{30}\}$

$= 10{,}000^2\ \{1.14506^{15} - 1.07^{30}\}$

$= 1{,}597{,}283.16$

SD (accumulation) $= \sqrt{1597283.16} = £1,263.84$

(iii) (a) By symmetry $j = 0.07$ (as in (i))

Hence, mean (accumulation) will be the same as in (i) (i.e. £27,590.32).

The spread of the yields around the mean is lower than in (i). Hence, the standard deviation of the accumulation will be lower than £1,263.84.

(b) Mean (accumulation) < £27,590.32 since the investment is being accumulated over a shorter period.

SD (accumulation) < £1,263.84 since investing over a shorter term than in (i) will lead to a narrower spread of possible accumulated amounts.

In part (i) some candidates misread the question and assumed the yield was fixed for the whole ten years rather than varying each year.

7 (i) Need $V_A(i) = V_L(i)$ with $i = 0.08$

$$V_L(i) = 6v^8 + 11v^{15}$$
$$V_A(\text{i}) = Xv^5 + Yv^{20}$$

Need $V_A'(i) = V_L'(i)$ with $i = 0.08$

$$V_L' = -48v^9 - 165v^{16}$$
$$V_A' = -5Xv^6 - 20Y\,v^{21}$$

Thus we have to solve simultaneous equations:

(a) $6v^8 + 11v^{15} = Xv^5 + Yv^{20}$

(b) $-48v^9 - 165v^{16} = -5Xv^6 - 20Y\,v^{21}$

Taking 5 times (a) + (1+i) times (b) we get

$$-18v^8 - 110v^{15} = -15Yv^{20}$$

$$\Rightarrow Y = \frac{18(1+i)^{12} + 110(1+i)^5}{15}$$

$$\Rightarrow Y = 13.79688$$

Substitute back in (a) to get $X = 5.50877$

Hence the values of the zero-coupon bonds are £5.50877 million and £13.79688 million.

(ii) We need to check that the third condition is satisfied:

$$V_A' = -5Xv^6 - 20Yv^{21}$$
$$\Rightarrow V_A'' = 30Xv^7 + 420Yv^{22}$$
$$\Rightarrow V_A''(0.08) = 30 \times 5.50877 \times 1.08^{-7} + 420 \times 13.79688 \times 1.08^{-22}$$
$$= 1162.31$$

$$V_L' = -48v^9 - 165v^{16}$$
$$\Rightarrow V_L'' = 432v^{10} + 2640v^{17}$$
$$\Rightarrow V_L''(0.08) = 432 \times 1.08^{-10} + 2640 \times 1.08^{-17}$$
$$= 913.61$$

Therefore $V_A''(0.08) > V_L''(0.08)$

Thus the third condition is satisfied.

[Or note that since the assets have terms of 5 years and 20 years and the liabilities have terms of 8 years and 15 years, the spread of assets around the mean term is greater than that of the liabilities. Hence, the convexity of assets is greater than the convexity of liabilities].

The best answered question on the paper.

8 (i) Work in £ millions

Let Discounted Payback Period from 1 January 2014 be n.

Then, considering project at the end of year n but before the outgo at the start of year n + 1

$$
\begin{aligned}
&-\ 19-9v^{1/2}-5v \\
&-\ 6\times 9.5\left(v^2+1.04v^3+\ldots+(1.04)^{n-3}v^{n-1}\right) \\
&+\ 6\times 12.6\left(v^3+1.04v^4+\ldots+(1.04)^{n-3}v^{n}\right)\geq 0 \quad \text{at } 9\%
\end{aligned}
$$

Hence, $19+8.6204+4.5872\leq\left(75.6v^3-57v^2\right)\left(\dfrac{1-\left(\dfrac{1.04}{1.09}\right)^{n-2}}{1-\dfrac{1.04}{1.09}}\right)$

and RHS = $10.4013\times 21.8\times\left(1-\left(\dfrac{1.04}{1.09}\right)^{n-2}\right)$

Hence, $\dfrac{32.2076}{10.4013\times 21.8}\leq 1-\left(\dfrac{1.04}{1.09}\right)^{n-2}$

$$\Rightarrow\left(\frac{1.04}{1.09}\right)^{n-2}\geq 0.85796$$

$$\Rightarrow(n-2)\log\left(\frac{1.04}{1.09}\right)\geq\ \log 0.85796$$

$$\Rightarrow n-2\geq\frac{-0.06653}{-0.02039}=3.262$$

$$\Rightarrow n\geq 5.262$$

But sales are only made at the end of each calendar year.

$\Rightarrow$ DPP = 6 years

(ii) The DPP would be shorter using an effective rate of interest less than 9% p.a. This is because the income (in the form of car sales) does not commence until a few years have elapsed whereas the bulk of the outgo occurs in the early years. The effect of discounting means that using a lower rate of interest has a greater effect on the value of the income than on the value of the outgo (although both values increase). Hence the DPP becomes shorter.

In part (i), many candidates valued the total outgo for the whole production run and then attempted to find when the present value of income exceeded this. The working of many marginal candidates was difficult to follow and it was not clear to the examiners what the candidates were attempting to do.

9 (i) $\frac{D}{R}(1-t_1) = \frac{0.08}{1} \times 0.7 = 0.056 < \underset{6\%}{i}^{(2)} = 0.059126$

$\Rightarrow$ There is a capital gain and assume redeemed as late as possible.

Let $P =$ Price at 1/5/11 per £100 nominal

$$P = \left[0.7 \times 8\, a_{\overline{11}|}^{(2)} + 100v^{11} - 0.25(100-P)v^{11}\right] \times (1+i)^{4/12}$$

$$\Rightarrow P = 5.6 \times 1.014782 \times 7.8869 \times (1.06)^{4/12} + 75v^{10\,8/12} + 0.25Pv^{10\,8/12}$$

$$\Rightarrow P = \frac{45.6985 + 40.2839}{1 - 0.25v^{10\,8/12}}$$

$$= £99.319$$

(ii) $\frac{D}{R} = 0.08 > i_{7\%}^{(2)} = 0.068816$

$\Rightarrow$ Assume redeemed as soon as possible

$$\text{Sale Price per £100 nominal} = \left(8\, a_{\overline{4}|}^{(2)} + 100v^4\right) \times (1+i)^{3/12}$$

$$= (8 \times 1.017204 \times 3.3872 + 100 \times 0.76290) \times (1.07)^{3/12}$$

$$= £\,105.625$$

(iii) CGT is payable of $(105.625 - 99.319) \times 0.25$

$= £1.5765$

Equation of value:

$$99.319 = 0.7\times 4\times v^{2/12} + 0.7\times 4\times v^{8/12} + 0.7\times 4\times v^{12/12} + 0.7\times 4\times v^{18/12} + (105.625 - 1.5765)v^{11/12}$$

$$\Rightarrow 99.319 = (1+i)^{4/12}\times 5.6\, a_{\overline{2|}}^{(2)} + 104.0485 v^{11/12}$$

At 8%, RHS is $1.08^{4/12}\times 5.6\times 1.019615\times 1.7833 + 104.0485v^{11/12}$
$= 100.226$

At 9% RHS is $(1.09)^{4/12}\times 5.6\times 1.022015\times 1.7591 + 104.0485v^{11/12}$
$= 98.568$

and since 98.568 < 99.319 < 100.226, the net yield is between 8% and 9% p.a.

Many candidates struggled with the four month adjustment in part (i). Common errors included:

- *ignoring the adjustment completely.*
- *discounting the present value of payments by four months rather than accumulating.*
- *adjusting the price at the end of the calculations (which does not allow for CGT correctly).*

In part (iii) some candidates wasted time by trying to solve the yield exactly rather than just show that 8% was too low and 9% too high.

10 (i) Original amount of loan is:

$$\begin{aligned} L &= 5,000v + 4,800v^2 + 4,600v^3 + \ldots + 1,200v^{20} \\ &= 5,200\times\left(v + v^2 + \ldots + v^{20}\right) - 200\times\left(v + 2v^2 + \ldots + 20v^{20}\right) \\ &= 5,200a_{\overline{20|}} - 200(Ia)_{\overline{20|}} \\ &= 5,200\times 13.5903 - 200\times 125.1550 \\ &= £45,638.56 \end{aligned}$$

(ii) Amount of 12th instalment is £2,800.

Loan o/s after 11th instalment is given by PV of future repayments:

$$\begin{aligned} L_{11} &= 2,800v + 2,600v^2 + 2,400v^3 + \ldots + 1,200v^9 \\ &= 3,000a_{\overline{9|}} - 200(Ia)_{\overline{9|}} \\ &= 3,000\times 7.4353 - 200\times 35.2366 \\ &= £15,258.58 \end{aligned}$$

Then, interest component of 12th instalment is: $0.04\times 15,258.58 = £610.34$.

Hence, capital repaid in 12th instalment is $2,800 - 610.34 = £2,189.66$.

(iii) (a) Then, after 12th instalment, loan o/s is

$15,258.58 - 2,189.66 = £13,068.92$.

This will be repaid by level instalments of £2,800.

Thus, remaining term of loan is n given by:

$$13,068.92 \le 2,800 \times a_{\overline{n}|}^{4\%} \Rightarrow a_{\overline{n}|}^{4\%} \ge 4.6675 \Rightarrow n = 6$$

i.e. remaining term is 6 years (i.e. loan is repaid by time 18)

(b) We need to find reduced final payment, R , such that:

$$13,068.92 = 2,800 \times a_{\overline{5}|}^{4\%} + Rv_{4\%}^{6} \Rightarrow 0.79031R$$
$$= 13,068.92 - 2,800 \times 4.4518 \Rightarrow R = £764.11$$

(c) Total amount of interest paid is given by:

$$5,000 + 4,800 + 4,600 + \ldots + 2,800 + 5 \times 2,800 + 764.11 - 45,638.56$$
$$= £15,925.55$$

In part (ii) the most common error was to not round n up, i.e. quoting a non-integer number of years for the revised loan. Part (iii) was answered poorly with candidates often not correctly allowing for the payments prior to the change in payment schedule.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

23 September 2013 (pm)

Subject CT1 – Financial Mathematics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 The rate of interest is 4.5% per annum effective.

(i) Calculate:

(a) the annual effective rate of discount.
(b) the nominal rate of discount per annum convertible monthly.
(c) the nominal rate of interest per annum convertible quarterly.
(d) the effective rate of interest over a five year period.

[5]

(ii) Explain why your answer to part (i)(b) is higher than your answer to part (i)(a). [2]

[Total 7]

2 A nine-month forward contract is issued on 1 March 2012 on a share with a price of £1.80 at that date. Dividends of 10p per share are expected on 1 September 2012.

Calculate the forward price at issue assuming a risk-free rate of interest of 4% per annum effective and no arbitrage. [3]

3 A fixed-interest security pays coupons of 4% per annum, half-yearly in arrear and will be redeemed at par in exactly ten years.

(i) Calculate the price per £100 nominal to provide a gross redemption yield of 3% per annum convertible half-yearly. [2]

(ii) Calculate the price, 91 days later, to provide a net redemption yield of 3% per annum convertible half-yearly if income tax is payable at 25%. [2]

[Total 4]

4 Describe the characteristics of the cash flows that are paid and received in respect of:

(i) an index-linked security. [2]
(ii) an equity. [3]

[Total 5]

5 An investor is considering the purchase of two government bonds, issued by two countries A and B respectively, both denominated in euro.

Both bonds provide a capital repayment of €100 together with a final coupon payment of €6 in exactly one year. The investor believes that he will receive both payments from the bond issued by Country A with certainty. He believes that there are four possible outcomes for the bond from Country B, shown in the table below.

Outcome	*Probability*
No coupon or capital payment	0.1
Capital payment received, but no coupon payment received	0.2
50% of capital payment received, but no coupon payment received	0.3
Both coupon and capital payments received in full	0.4

The price of the bond issued by Country A is €101.

(i) Calculate the price of the bond issued by Country B to give the same expected return as that for the bond issued by Country A. [3]

(ii) Calculate the gross redemption yield from the bond issued by Country B assuming that the price is as calculated in part (i). [1]

(iii) Explain why the investor might require a higher expected return from the bond issued by Country B than from the bond issued by Country A. [2]

[Total 6]

6 A pension fund is considering investing in a major infrastructure project. The fund has been asked to make an investment of £2m for a 1% share in revenues from building a road. No other costs will be incurred by the pension fund. The following revenues are expected to arise from the project:

In the first year, 40,000 vehicles a day will use the road, each paying a toll of £1.

In the second year, 50,000 vehicles a day will use the road, each paying a toll of £1.10.

In the third year, both the number of vehicles using the road and the level of tolls will rise by 1% from their level in the second year. They will both continue to rise by 1% per annum compound until the end of the 20th year.

At the end of the 20th year, it is assumed that the road has no value as it will have to be completely rebuilt.

You should assume that all revenue is received continuously throughout the year and that there are 365 days in all years.

Calculate the net present value of the investment in the road at a rate of interest of 8% per annum effective. [10]

7 An insurance company has just written contracts that require it to make payments to policyholders of £10 million in five years' time. The total premiums paid by policyholders at the outset of the contracts amounted to £7.85 million. The insurance company is to invest the premiums in assets that have an uncertain return. The return from these assets in year t, i_t, has a mean value of 5.5% per annum effective and a standard deviation of 4% per annum effective. $(1+i_t)$ is independently and lognormally distributed.

(i) Calculate the mean and standard deviation of the accumulation of the premiums over the five-year period. You should derive all necessary formulae. [Note: You are not required to derive the formulae for the mean and variance of a lognormal distribution.] [9]

A director of the insurance company is concerned about the possibility of a considerable loss from the investment strategy suggested in part (i). He therefore suggests investing in fixed-interest securities with a guaranteed return of 4 per cent per annum effective.

(ii) Explain the arguments for and against the director's suggestion. [3]
[Total 12]

8 Mrs Jones invests a sum of money for her retirement which is expected to be in 20 years' time. The money is invested in a zero coupon bond which provides a return of 5% per annum effective. At retirement, the individual requires sufficient money to purchase an annuity certain of £10,000 per annum for 25 years. The annuity will be paid monthly in arrear and the purchase price will be calculated at a rate of interest of 4% per annum convertible half-yearly.

(i) Calculate the sum of money the individual needs to invest at the beginning of the 20-year period. [5]

The index of retail prices has a value of 143 at the beginning of the 20-year period and 340 at the end of the 20-year period.

(ii) Calculate the annual effective real return the individual would obtain from the zero coupon bond. [2]

The government introduces a capital gains tax on zero coupon bonds of 25 per cent of the nominal capital gain.

(iii) Calculate the net annual effective real return to the investor over the 20-year period before the annuity commences. [3]

(iv) Explain why the investor has achieved a negative real rate of return despite capital gains tax only being a tax on the profits from an investment. [2]
[Total 12]

9 A bank makes a loan to be repaid by instalments paid annually in arrear. The first instalment is £400, the second is £380 with the payments reducing by £20 per annum until the end of the 15^{th} year, after which there are no further repayments. The rate of interest charged is 4% per annum effective.

(i) Calculate the amount of the loan. [3]

(ii) Calculate the capital and interest components of the first payment. [2]

At the beginning of the ninth year, the borrower can no longer make the scheduled repayments. The bank agrees to reduce the capital by 50 per cent of the loan outstanding after the eighth repayment. The bank requires that the remaining capital is repaid by a 10-year annuity paid annually in arrear, increasing by £2 per annum. The bank changes the rate of interest to 8% per annum effective.

(iii) Calculate the first repayment under the revised loan. [5]
[Total 10]

10 The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t) = 0.05 + 0.002t$$

Calculate the accumulated value of a unit sum of money:

(i) (a) accumulated from time $t = 0$ to time $t = 7$.
(b) accumulated from time $t = 0$ to time $t = 6$.
(c) accumulated from time $t = 6$ to time $t = 7$.
[5]

(ii) Calculate, using your results from part (i) or otherwise:

(a) the seven-year spot rate of interest per annum from time $t = 0$ to time $t = 7$.

(b) the six-year spot rate of interest per annum from time $t = 0$ to time $t = 6$.

(c) $f_{6,1}$ where $f_{6,1}$ is the one-year forward rate of interest per annum from time $t = 6$. [3]

(iii) Explain why your answer to part (ii)(c) is higher than your answer to part (ii)(a). [2]

(iv) Calculate the present value of an annuity that is paid continuously at a rate of $30e^{-0.01t+0.001t^2}$ units per annum from $t = 3$ to $t = 10$. [5]
[Total 15]

11 A pension fund has liabilities to meet annuities payable in arrear for 40 years at a rate of £10 million per annum.

The fund is invested in two fixed-interest securities. The first security pays annual coupons of 5% and is redeemed at par in exactly ten years' time. The second security pays annual coupons of 10% and is redeemed at par in exactly five years' time. The present value of the assets in the pension fund is equal to the present value of the liabilities of the fund and exactly half the assets are invested in each security. All assets and liabilities are valued at a rate of interest of 4% per annum effective.

(i) Calculate the present value of the liabilities of the fund. [1]

(ii) Calculate the nominal amount held of each security purchased by the pension fund. [6]

(iii) Calculate the duration of the liabilities of the pension fund. [3]

(iv) Calculate the duration of the assets of the pension fund. [4]

(v) Without further calculations, explain whether the pension fund will make a profit or loss if interest rates fall uniformly by 1.5% per annum effective. [2]
[Total 16]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2013 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

December 2013

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2013 paper

This paper proved to have some questions where the vast majority of candidates scored well and others where many candidates found challenging. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q1(ii), Q8(iv), Q10(iii) and Q11(v) were less well answered than those that just involved calculation. This is an area to which attention should be paid. Candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) (a) $d = \dfrac{0.045}{1.045} = 0.043062 = 4.3062\%$

(b) $\left(1 - d^{(12)}/_{12}\right) = (1.045)^{-1/12}$

$\therefore 1 - \dfrac{d^{(12)}}{12} = 0.99634$

$\therefore d^{(12)} = 0.043936$ or 4.3936%

(c) $\left(1 + \dfrac{i^{(4)}}{4}\right)^4 = 1.045$

$\therefore \left(1 + \dfrac{i^{(4)}}{4}\right) = 1.011065$

$\therefore i^{(4)} = 0.044260$ or 4.4260%

(d) $1.045^5 = 1.24618$

$\therefore$ five-yearly effective rate is 24.618%

(ii) The answer to (i)(b) is bigger than the answer to (i)(a) because the rate of discount convertible monthly is applied each month to a smaller (already discounted) sum of money. As such, in order to achieve the same total amount of discounting the rate has to be slightly more than one twelfth of the annual rate of discount. [An answer relating to the concept of interest payable in advance would also be acceptable].

The calculations were performed well but the quality of the explanations in part (ii) was often poor. A common error in (i)(d) was to state the answer as $i^{(1/5)}$ *rather than* $\dfrac{i^{(1/5)}}{1/5}$.

2 Present value of dividend = $0.1 \times 1.04^{-0.5} = 0.09806$

Value of forward is $(1.8 - 0.09806) \times 1.04^{0.75} = £1.75275$

3 (i) Work in half years.

$$P = 2a_{\overline{20}|} + 100v^{20} \text{ @ } 1\tfrac{1}{2}\%$$
$$= 2\times 17.1686 + 100\times 0.74247$$
$$= £108.584$$

(ii) $$P = (2\times 0.75a_{\overline{20}|} + 100v^{20})(1.015)^{91/182.5}$$
$$= (2\times 0.75\times 17.1686 + 100\times 0.74247)\times (1.015)^{91/182.5}$$
$$= £100.7452$$

Part (i) was answered well although some candidates assumed an annual effective rate of 3%. In part (ii) many candidates did not deal with the 91 days elapsed duration – discounting instead of accumulating the 10-year bond price and/or assuming that 91 days equated to a quarter of a year.

4 (i) The investor pays a purchase price at outset.

The investor receives a series of coupon payments and a capital payment at maturity

The coupon and capital payments are linked to an index of prices (possibly with a time lag)

[Time lag does not have to be mentioned].

(ii) The investor pays a purchase price at outset

Shareholders are paid dividends. These are not fixed but declared out of profits.
Dividends may be expected to increase over time ….
….but may cease if the company fails.

There is a high degree of uncertainty with regard to future cash flows.

No maturity date
Would receive a sale price on the sale of the shares

Generally poorly answered with many candidates just writing down all characteristics they knew about these assets rather than concentrating on the cashflows. Many candidates omitted mention of the initial purchase price in each part.

5 (i) The return from the bonds issued by Country A is: $\frac{106}{101} - 1 = 0.049505$

The expected cash flows from the bonds from Country B are:

$$0.1 \times 0 + 0.2 \times 100 + 0.3 \times 50 + 0.4 \times 106 = 77.4$$

The price to provide the same expected return is P such that:

$$P = \frac{77.4}{1.049505} = €73.749$$

(ii) The gross redemption yield from the bond is such that:

$$73.749 \times (1+i) = 106$$
$$\therefore i = 43.731\%$$

(iii) The risk is higher for Country B's bond. Although the gross redemption yield is such that the expected returns are equal, the investor may want a higher expected return to compensate for the higher risk.

Many candidates had trouble with part (ii) not recognising that the gross redemption yield calculation will not include any allowance for default.

6 Divide the number of cars by 100 to obtain the share due to the pension fund

PV of income = $365 \times 400\, \bar{a}_{\overline{1}|} + 365 \times 500\, \bar{a}_{\overline{1}|}\, v \times 1.1 \times \left(1 + 1.01^2 v + 1.01^4 v^2 + \ldots\ldots + 1.01^{36} v^{18}\right)$

$$= 365 \times 400 \frac{i}{\delta} a_{\overline{1}|} + 365 \times 500 \frac{i}{\delta} a_{\overline{1}|}\, v \times 1.1 \times \left(\frac{1 - 1.01^{38} v^{19}}{1 - 1.01^2 v}\right)$$

$$= 365 \times 400 \times 1.039487 \times 0.92593$$
$$+ 365 \times 500 \times 1.039487 \times 0.92593 \times 0.92593 \times 1.1 \times \left(\frac{1 - 1.45953 \times 0.23171}{1 - 1.0201 \times 0.92593}\right)$$

$$= 140{,}523 + 178{,}907 \times 11.93247$$

$$= 140{,}523 + 2{,}134{,}801 = 2{,}275{,}324 \text{ so NPV} = £275{,}324$$

Candidates made a variety of errors in this question often ignoring one or more parts of the scenario (e.g. pension fund's 1% share of the project, the fact that daily vehicle numbers were given in the question, 1% increases in both vehicle numbers and tolls from the second year). Nevertheless, candidates who set out their workings clearly and logically often scored the majority of the available marks.

7 (i) $(1 + i_t) \sim$ lognormal (μ, σ^2)

$$\ln(1+i_t) \sim N(\mu, \sigma^2)$$

$$\ln \prod_{t=1}^{5} (1+i_t) = \ln(1+i_t) + \ln(1+i_t) + \ldots + \ln(1+i_t)$$

$$\sim N(5\mu, 5\sigma^2) \quad \text{by independence}$$

$$\therefore \prod_{t=1}^{5} (1+i_t) \sim \text{ lognormal } (5\mu, 5\sigma^2)$$

$$E(1+i_t) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = 1.055$$

$$\text{Var}(1+i_t) = \exp(2\mu + \sigma^2)\left[\exp(\sigma^2) - 1\right] = 0.04^2$$

$$\frac{0.04^2}{1.055^2} = \left[\exp\left(\sigma^2\right) - 1\right] \therefore \sigma^2 = 0.0014365$$

$$\exp\left(\mu + \frac{0.0014360}{2}\right) = 1.055 \Rightarrow$$

$$\mu = \ln 1.055 - \frac{0.0014365}{2} = 0.052823$$

$5\mu = 0.264113$

$5\sigma^2 = 0.007182.$

Let S_5 be the accumulation of one unit after five years:

$$E(S_5) = \exp\left(5 \times \mu + \frac{5\sigma^2}{2}\right) = \exp\left(0.264113 + \frac{0.007182}{2}\right)$$

$$= 1.30696$$

$$\text{Var}(S_5) = \exp(2 \times 5\mu + 5\sigma^2)\left[\exp\left(5\sigma^2\right) - 1\right]$$

$$= \exp(2 \times 0.264113 + 0.007182).(\exp 0.007182 - 1)$$

$= \exp 0.53541 (\exp 0.007182 - 1)$

$= 0.012313$

Mean value of the accumulation of premiums is

$7{,}850{,}000 \times 1.30696 = £10{,}259{,}636.$

Standard deviation of the accumulated value of the premiums is

$$7{,}850{,}000 \times \sqrt{0.012313} = £871{,}061$$

Alternatively:

Let i_t be the (random) rate of interest in year t. Let S_5 be the accumulation of a single investment of 1 unit after five years:

$$E(S_5) = E\left[(1+i_1)(1+i_2)\ldots(1+i_5)\right]$$

$$E(S_5) = E[1+i_1]E[1+i_2]\ldots E[1+i_5] \text{ as } \{i_t\} \text{ are independent}$$

$$E[i_t] = 0.055$$

$$\therefore \quad E(S_5) = (1.055)^5 = 1.30696$$

$$E\left(S_5^2\right) = E\left[\left[(1+i_1)(1+i_2)\ldots(1+i_5)\right]^2\right]$$

$$= E(1+i_1)^2 E(1+i_2)^2 \ldots E(1+i_5)^2 \text{ (using independence)}$$

$$= E\left(1+2i_1+i_1^2\right)E\left(1+2i_2+i_2^2\right)\ldots E\left(1+2i_5+i_5^2\right)$$

$$= \left(1+2\times 0.055+0.04^2+0.055^2\right)^5$$

as $E\left[i_i^2\right] = V[i_t] + E[i_t]^2 = 0.04^2 + 0.055^2$

$$\therefore \ \mathrm{Var}[S_5] = \left(1+2\times 0.055+0.04^2+0.055^2\right)^5 - (1.055)^{10}$$

$$\therefore \ = 1.114625^5 - (1.055)^{10} = 0.0123128$$

Mean value of the accumulation of premiums is

$$7{,}850{,}000 \times 1.30696 = £10{,}259{,}636.$$

Standard deviation of the accumulated value of the premiums is

$$7{,}850{,}000 \times \sqrt{0.012313} = £871{,}061$$

(ii) If the company invested in fixed-interest securities, it would obtain a guaranteed accumulation of £7,850,000 $(1.04)^5$ = £9,550,725. In one sense, there is a 100% probability that a loss will be made and therefore the policy is unwise. The "risky" investment strategy leads to an expected profit. On the other hand, the standard deviation of the accumulation from the risky investment strategy is £871,061. Whilst there is a chance of an even greater profit from this strategy, there is also a chance of a more considerable loss than from investing in fixed-interest securities.

A poorly answered questions with many candidates not including enough derivation of the required results in part (i). Some candidates mixed their answers between the two methods given above e.g. they calculated μ and σ^2 for the log normal route, then used these in the alternative method for the mean and variance of i_t. Other candidates just used 0.055 and $.04^2$ as their values of μ and σ^2 .

8 (i) Purchase price of the annuity (working in half-years)

$$5{,}000 a_{\overline{50}|}^{(6)} \text{ calculated at } i = 2\%$$

$$= 5{,}000 \frac{i}{i^{(6)}} a_{\overline{50}|}$$

$i = 0.02$

$i^{(6)} = 0.019835$

$a_{\overline{50}|} = 31.4236$

$$\text{Purchase price} = 5{,}000 \times \frac{0.02}{0.019835} \times 31.4236$$

$$= £158{,}422$$

Individual needs to invest X such that: (working in years)

$$X\,1.05^{20} = 158{,}422$$

$$1.05^{20} = 2.653297$$

$$\therefore X = \frac{158{,}422}{2.653297} = £59{,}708$$

(ii) Real return is j such that:

$$59{,}708 = \frac{158{,}422}{(1+j)^{20}} \times \frac{143}{340}$$

$$\therefore (i+j)^{20} = \frac{158{,}422}{59{,}708} \times \frac{143}{340}$$

$$= 1.11595$$

$\therefore j = 0.550\%$

(iii) The amount of the capital gain is:

$$158{,}422 - 59{,}708 = 98{,}714$$

Tax = 0.25 × 98,714 = 24,679

Proceeds of investment = 133,744

Net real return is j' such that:

$$59{,}708 = \frac{133{,}744}{(1+j')^{20}} \times \frac{143}{340}$$

$$\therefore (1+j')^{20} = \frac{133{,}744}{59{,}708} \times \frac{143}{340}$$

$$= 0.942106$$

$\therefore\ j' = -0.2977\%$

(iv) The capital gains taxed has taxed the nominal gain, part of which is merely to compensate the investor for inflation. The tax has therefore reduced the real value of the investor's capital and led to a negative real return.

Parts (i) and (ii) were generally answered well but many candidates struggled with the calculation of the capital gain in part (iii) not recognising that this would be based on money values.

9 (i) PV is:

$$400v + 380v^2 + 360v^3 + \cdots + 120v^5$$

$$420\, a_{\overline{15}|} - 20\left(Ia\right)_{\overline{15}|} \qquad @\,4\%$$

= 420 × 11.1184 - 20 × 80.8539

= 4,669.728 – 1,617.078 = £3,052.65.

(ii) Interest component:

= 0.04 × 3,052.65 = £122.106

Capital component = 400 – 122.106
= £277.894

(iii) Seven repayments remain and the PV of the remaining payments is:

$$240v + 220v^2 + \cdots + 120\,v^7$$

$$260\, a_{\overline{7}|} - 20\left(Ia\right)_{\overline{7}|} \qquad @\,4\%$$

$$= 260 \times 6.0021 - 20 \times 23.0678 = £1,099.19$$

The loan is written down to: 0.5 × 1,099.19
= £549.595

The present value of the new repayment is:

$$\left(X - 2\right) a_{\overline{10}|} + 2(Ia)_{\overline{10}|} \qquad @\,8\%$$

$$\therefore 549.595 = \left(X - 2\right)6.7101 + 2 \times 32.6869$$

$$\therefore X - 2 = \frac{549.595 - 2 \times 32.6869}{6.7101} = 72.163$$

$$\therefore X = £74.16$$

The best answered question on the paper although some candidates, when calculating the outstanding loan in part (iii), stated that the repayment in year 8 was £420. Some candidates also used the incorrect formula $Xa_{\overline{10}|} + 2(Ia)_{\overline{10}|}$ *for the repayment in part (iv).*

10 (i) (a) $e^{\int_0^7 0.05+0.002t\,dt}$

$= e^{\left[0.05t+\frac{0.002t^2}{2}\right]_0^7}$

$= \exp\left[0.05\times 7\right]+\dfrac{0.002\times 49}{2}$

$= \exp(0.399) = 1.490331$

(b) $e^{\int_0^6 0.05+0.002t\,\mathrm{dt}}$

$= e^{\left[0.05\times 6+\frac{0.002\times 36}{2}\right]}$

$= \exp(0.336) = 1.399339$

(c) $\dfrac{1.490331}{1.399339} = 1.06503$

(ii) (a) Let spot rate = i_7

$(1+i_7)^7 = 1.490334$

$\Rightarrow i_7 = 5.8656\%$ p.a. effective

(b) $(1+i_6)^6 = 1.39934$

$\therefore i_6 = 5.7598\%$ p.a. effective

(c) From (i) (c) 6.503% per annum effective.

(iii) The forward rate is the rate of interest in the seventh year. The spot rate, in effect, is the rate of interest per annum averaged over the seven years (a form of geometric average). As the force of interest is rising the rate of interest in the seventh year must be higher than the rate averaged over the full seven year period.

(iv) $v(t) = e^{-\int_0^t 0.05+0.002s \, ds}$

$$= e^{-\left[0.05s+\frac{0.002s^2}{2}\right]_0^t}$$

$$= e^{-0.05t-0.001t^2}$$

We require

$$\int_3^{10} \rho(t)v(t)dt$$

$$= \int_3^{10} \frac{30\ \mathrm{e}^{-0.01t}}{e^{-0.001t^2}}.e^{-0.05t}.e^{-0.001t^2}dt$$

$$30\int_3^{10} e^{-0.06t}\ dt$$

$$\frac{30}{-0.06}\left[e^{-0.06t}\right]_3^{10}$$

$$= \frac{30}{-0.06}\left[e^{-0.6}-e^{-0.18}\right]$$

$= -500(0.548812 - 0.83527)$

$=143.229$

The calculations were well-done but only the best candidates clearly explained their reasoning in part (iii).

11 (i) Present value of liabilities annuity

$10\ a_{\overline{40|}}$ at 4% $\quad a_{\overline{40|}} = 19.7928$

$= 10 \times 19.7928 = £197.928$m

(ii) Call 10 year security "security A" and five year security "security B".

We need to calculate the PV of £100 nominal for each of security A and security B

P.V of £100 nominal of A is:

$$5a_{\overline{10}|} + 100v^{10} \quad @4\%$$

$$a_{\overline{10}|} = 8.1109 \quad v^{10} = 0.67556$$

$\therefore$ PV = 5×8.1109 + 67.556 = 108.1105

P.V of £100 nominal of B is:

$$10\ a_{\overline{5}|} + 100v^{5} \quad @\,4\%$$

$$a_{\overline{5}|} = 4.4518 \quad v^{5} = 0.82193$$

$\therefore$ PV = 44.518 + 82.193 = 126.711

£98.964m, is invested in each security.

$$\frac{98,964,000}{108.1105} \times 100 \text{ per £100 nominal of A is bought.}$$

= £91,539,674 nominal

$$\frac{98,964,000}{126.711} \times 100 \text{ per £100 nominal of B is bought}$$

= £78,102,138 nominal

[other ways of expressing units are okay, but marks will be deducted if units are not correct]

(iii) Duration of the liabilities

$$= \frac{\sum tc_t \ v^t}{\sum c_t \ v^t}$$

$$\text{Numerator} = \sum_{t=1}^{40} 10\ t\ v^t \quad \text{(in £ m)}$$

$$= 10(Ia)_{\overline{40}|} = 10 \times 306.3231 = 3063.231 \quad \text{at 4\% p.a. effective}$$

$\therefore$ Duration = 3063.231/197.928 = 15.48 years

(iv) Numerator of duration is:

$$(5(Ia)_{\overline{10|}} + 10 \times 100\, v^{10}) \times 915{,}396.74$$

$$+(10(Ia)_{\overline{5|}} + 5 \times 100\, v^{5}) \times 781{,}021.38$$

Following the same reasoning as for the calculation of the duration of the annuity payments, adding the capital repayment and multiplying by the number of units of £100 nominal bought.

$$(Ia)_{\overline{10|}} = 41.9922$$

$$v^{10} = 0.67556$$

$$(Ia)_{\overline{5|}} = 13.0065$$

$$v^{5} = 0.82193$$

$$= (5 \times 41.9922 + 10 \times 100 \times 0.67556) \times 915{,}396.74$$

$$+(10 \times 13.0065 + 5 \times 100 \times 0.82193) \times 781{,}021.38$$

$$= 810{,}603{,}000 + 422{,}554{,}000$$

$$= 1{,}233{,}157{,}000$$

$\therefore$ Duration = 1,233,157,000/197,928,000
= 6.23 years

(v) The duration (and therefore the volatility) is greater for the liabilities than for the assets. As a result, when interest rates fall, the present value of the liabilities will rise by more than the present value of the assets and so a loss will be made.

Many candidates wrongly assumed that the same nominal amounts were bought of each asset rather than each asset amount having the same present value. This assumption made the calculations in part (ii) somewhat easier and the marks awarded in this part took this into account. Part (iii) was answered well. The explanations in part (v) were often poorly stated although time pressures at the end of the paper may have contributed to this.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

22 April 2014 (am)

Subject CT1 – Financial Mathematics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 12 questions, beginning your answer to each question on a new page.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 You are given the following information in respect of a pension fund:

Calendar Year	*Value of fund at 1 January*	*Value of fund at 30 June*	*Net cash flow received on 1 July*
2011	£870,000	£872,000	£26,000
2012	£914,000	£902,000	£27,000
2013	£953,000	£962,000	£33,000
2014	£990,000		

Calculate, to the nearest 0.1%, the annual effective money-weighted rate of return earned by the fund during the period from 1 January 2011 to 1 January 2014. [4]

2 Describe the main features of:

(a) debenture stocks.
(b) unsecured loan stocks. [5]

3 £900 accumulates to £925 in four months.

Calculate the following:

(i) the nominal rate of interest per annum convertible half-yearly [2]
(ii) the nominal rate of discount per annum convertible quarterly [2]
(iii) the simple rate of interest per annum [2]
[Total 6]

4 A company issues a loan stock bearing interest at a rate of 8% per annum payable half-yearly in arrear. The stock is to be redeemed at 103% on any coupon payment date in the range from 20 years after issue to 25 years after issue inclusive, to be chosen by the company.

An investor, who is liable to income tax at 30% and tax on capital gains at 40%, bought the stock at issue at a price which gave her a minimum net yield to redemption of 6% per annum effective.

Calculate the price that the investor paid. [7]

5 On 25 October 2008 a certain government issued a 5-year index-linked stock. The stock had a nominal coupon rate of 3% per annum payable half-yearly in arrear and a nominal redemption price of 100%. The actual coupon and redemption payments were index-linked by reference to a retail price index as at the month of payment.

An investor, who was not subject to tax, bought £10,000 nominal of the stock on 26 October 2012. The investor held the stock until redemption.

You are given the following values of the retail price index:

	2008	-----	2012	2013
April	-----	-----	-----	171.4
October	149.2	-----	169.4	173.8

(i) Calculate the coupon payment that the investor received on 25 April 2013 and the coupon and redemption payments that the investor received on 25 October 2013. [3]

(ii) Calculate the purchase price that the investor paid on 25 October 2012 if the investor achieved an effective real yield of 3.5% per annum effective on the investment. [4]

[Total 7]

6 An insurance company has liabilities of £10 million due in 10 years' time and £20 million due in 15 years' time. The company's assets consist of two zero-coupon bonds. One pays £7.404 million in 2 years' time and the other pays £31.834 million in 25 years' time. The current interest rate is 7% per annum effective.

(i) Show that Redington's first two conditions for immunisation against small changes in the rate of interest are satisfied for this insurance company. [6]

(ii) Calculate the present value of profit that the insurance company will make if the interest rate increases immediately to 7.5% per annum effective. [2]

(iii) Explain, without any further calculation, why the insurance company made a profit as a result of the change in the interest rate. [2]

[Total 10]

7 Six months ago, an investor entered into a one-year forward contract to purchase a non-dividend paying stock. The risk-free force of interest was 4% per annum. The value of the stock is now 98% of its original value.

Calculate the minimum value for the risk-free force of interest at which the original forward contract still has a positive value to the investor. [6]

8 An insurance company borrows £50 million at an effective interest rate of 9% per annum. The insurance company uses the money to invest in a capital project that pays £6 million per annum payable half-yearly in arrear for 20 years. The income from the project is used to repay the loan. Once the loan has been repaid, the insurance company can earn interest at an effective interest rate of 7% per annum.

(i) Calculate the discounted payback period for this investment. [4]

(ii) Calculate the accumulated profit the insurance company will have made at the end of the term of the capital project. [5]

[Total 9]

9 The effective n-year spot rate of interest y_n, is given by:

$$y_n = 0.035 + \frac{n}{1000} \quad \text{for } n = 1, 2 \text{ and } 3$$

(i) Determine the implied one-year forward rates applicable at times $t = 1$ and $t = 2$ to four significant figures. [4]

(ii) Calculate, assuming no arbitrage:

(a) The price at time $t = 0$ per £100 nominal of a bond which pays annual coupons of 4% in arrear and is redeemed at 105% per £100 nominal after three years.

(b) The two-year par yield.

[6]

[Total 10]

10 A loan of £20,000 is repayable by an annuity payable annually in arrear for 25 years. The annual repayment is calculated at an effective interest rate of 8% per annum and increases by £50 each year.

(i) Calculate the amount of the first payment. [3]

(ii) Calculate the capital outstanding after the first three payments have been made. [2]

(iii) Explain your answer to part (ii). [2]

(iv) Calculate the total amount of interest paid over the term of the loan. [3]

[Total 10]

11 An individual can obtain a force of interest per annum at time t, measured in years, as given by the formula:

$$\delta(t) = \begin{cases} 0.03 + 0.01t & 0 \le t < 4 \\ 0.07 & 4 \le t < 6 \\ 0.09 & 6 \le t \end{cases}$$

(i) Calculate the amount the individual would need to invest at time $t = 0$ in order to receive a continuous payment stream of $3,000 per annum from time $t = 4$ to $t = 10$. [6]

(ii) Calculate the equivalent constant annual effective rate of interest earned by the individual in part (i). [3]
[Total 9]

12 An investor is considering investing £18,000 for a period of 12 years. Let i_t be the effective rate of interest in the t^{th} year, $t \le 12$. Assume, for $t \le 12$, that i_t has mean value of 0.08 and standard deviation 0.05 and that $1 + i_t$ is independently and lognormally distributed.

(i) Determine the distribution of S_{12} where S_t is the accumulation of £1 over t years. [5]

At the end of the 12 years the investor intends to use the accumulated amount of the investment to purchase a 12-year annuity certain paying:

£4,000 per annum monthly in advance during the first four years;
£5,000 per annum quarterly in advance during the second four years;
£6,000 per annum continuously during the final four years.

The effective rate of interest will be 7% per annum in years 13 to 18 and 9% per annum in years 19 to 24 where the years are counted from the start of the initial investment

(ii) Calculate the probability that the investor will meet the objective. [12]
[Total 17]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2014 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

June 2014

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2014 paper

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates

1 We can ignore the fund values given at 30 June.

Working in £000s:

$$870(1+i)^3 + 26(1+i)^{2\frac{1}{2}} + 27(1+i)^{1\frac{1}{2}} + 33(1+i)^{\frac{1}{2}} = 990$$

Approximate i comes from:

$$(870+26+27+33)(1+i)^3 = 990$$

$$\Rightarrow i = 1.2\%$$

Try 1%, LHS = 983.587

Try 2%, LHS = 1011.713

So

$$i = 0.01 + (0.02-0.01) \times \frac{990-983.587}{1011.713-983.587}$$

$$= 0.0123$$

Answer = 1.2% p.a.

Well answered although many candidates ignored the instruction to give the answer to the nearest 0.1%, and were penalised accordingly.

2 (a) **Debentures**

Debentures are part of the loan capital of companies.
The term "loan capital" usually refers to long-term borrowings rather than short-term.
Payments consist of regular coupons…
…and a final redemption payment
The issuing company provides some form of security to holders of the debenture…
…e.g. via a fixed or floating charge on the company's assets
Debenture stocks are considered more risky than government bonds…
…and are considered less marketable than government bonds.
Accordingly the yield required by investors will be higher than for a comparable government bond.

(b) **Unsecured loan stocks**

Issued by various companies.
They are unsecured – holders rank alongside other unsecured creditors.
Yields will be higher than on comparable debentures issued by the same company…
…to reflect the higher default risk.

This question was poorly answered despite being completely based on bookwork.

The above shows the variety of points that could be made (and not all were required for full marks). Many marginal candidates either made no significant attempt at the question or did not make enough distinct points.

3 (i) $$900 \times \left(1 + \frac{i^{(2)}}{2}\right)^{2*\frac{4}{12}} = 925$$

$$\Rightarrow \left(1 + \frac{i^{(2)}}{2}\right)^{\frac{8}{12}} = \frac{925}{900} \Rightarrow 1 + \frac{i^{(2)}}{2} = \left(\frac{925}{900}\right)^{\frac{12}{8}} = 1.041954693$$

$$\Rightarrow i^{(2)} = 8.39\% \quad (8.3909385)$$

(ii) $$900 = 925 \times \left(1 - \frac{d^{(4)}}{4}\right)^{4*\frac{4}{12}}$$

$$\Rightarrow \left(1 - \frac{d^{(4)}}{4}\right)^{\frac{16}{12}} = \frac{900}{925} \Rightarrow 1 - \frac{d^{(4)}}{4} = \left(\frac{900}{925}\right)^{\frac{12}{16}} = 0.979660466$$

$$\Rightarrow d^{(4)} = 8.14\% \quad (8.1358136)$$

(iii) $$900 \times \left(1 + \frac{4}{12} i'\right) = 925 \Rightarrow i' = 8.33\% \left(8.\dot{3}\right)$$

Where i' is the simple rate of interest per annum.

This question was answered very well although some candidates calculated $i^{(4)}$ rather than $d^{(4)}$ for part (ii).

4 Firstly we must consider $i^{(2)}$ and $(1-t)\dfrac{D}{R}$

where $i^{(2)}$ is evaluated at the net yield rate (6% p.a.) = 5.9126%

$t = 0.30$, the income tax rate

$$\frac{D}{R} = \frac{8}{1.03} = 7.7670 \text{ p.a.} \quad \Rightarrow (1-t)\frac{D}{R} = 5.4369\%$$

We have $i^{(2)} > (1-t)\dfrac{D}{R}$

$\Rightarrow$ there is a capital gain and the stock will be redeemed at the last possible date if the minimum yield is received. i.e. at the end of 25 years.
Hence, let P be price per £100 nominal, then

$$P = (1-0.3)8\, a_{\overline{25|}}^{(2)} + \left(103-(103-P)\times 0.4\right)v^{25} \text{at 6\% p.a.}$$

$$= 5.6a_{\overline{25|}}^{(2)} + (61.8+0.4P)v^{25}$$

$$\Rightarrow P = \frac{5.6\dfrac{i}{i^{(2)}}\, a_{\overline{25|}} + 61.8v^{25}}{1-0.4v^{25}}$$

$$= \frac{5.6\times 1.014782\times 12.7834 \ + 61.8\times 0.23300}{1-0.4\times 0.23300}$$

$$= \frac{72.6452+14.3994}{1-0.0932}$$

= £95.99

Generally well-answered although some candidates' arguments for choosing the latest possible date were unclear.

5 (i) The amounts of cash flows:

Coupon on 25/4/2013

$$= 10,000 \times \frac{0.03}{2} \times \frac{RPI_{4/2013}}{RPI_{10/2008}}$$

$$= 10000 \times \frac{0.03}{2} \times \frac{171.4}{149.2} = £172.319$$

Coupon on 25/10/2013

$$= 10000 \times \frac{0.03}{2} \times \frac{RPI_{10/2013}}{RPI_{10/2008}}$$

$$= 10000 \times \frac{0.03}{2} \times \frac{173.8}{149.2} = £174.732$$

$$\text{Redemption on 25/10/2013} = 10000 \times \frac{173.8}{149.2}$$

$$= £11,648.794$$

(ii) Purchase Price at 25/10/2012 = PV at real rate of $3\frac{1}{2}\%$ p.a. effective of future cash flows.

= PV at $3\frac{1}{2}\%$ p.a. effective of "25/10/2012 money values" of future cash flows.

Future cash flows expressed in 25/10/2012 money values

$$\text{Coupon at 25/4/2013} = 172.319 \times \frac{RPI_{10/2012}}{RPI_{4/2013}}$$

$$= 172.319 \times \frac{169.4}{171.4} = £170.308$$

$$\text{Coupon at 25/10/2013} = 174.732 \times \frac{RPI_{10/2012}}{RPI_{10/2013}}$$

$$= 174.732 \times \frac{169.4}{173.8} = £170.308$$

(same as 25/4/2013, as expected)

Redemption at 25/10/2013 = $11648.794 \times \dfrac{169.4}{173.8} \quad = \quad £11{,}353.888$

$$\left[\text{or } 10000 \times \frac{RPI_{10/2012}}{RPI_{10/2008}} = 10000 \times \frac{169.4}{149.2} = 11353.888 \right]$$

Hence Price at 25/10/2012

$$= 170.308 \times \frac{1}{(1.035)^{\frac{1}{2}}} + \frac{170.308 + 11353.888}{(1.035)}$$

$$= £11{,}301.89$$

Many candidates had difficulty in recognising that the real yield would be based on using the inflation-adjusted cashflows as at the time of purchase. Some candidates made no adjustment at all whereas others incorrectly assumed that the inflation rate would be constant throughout the holding period.

6 (i) Redington's first condition states that the *pv* of the assets should equal the *pv* of the liabilities.

Working in £ million:

pv of assets
$$\begin{aligned} &= 7.404v^2 + 31.834v^{25} \text{ at } 7\% \\ &= 7.404 * 0.87344 + 31.834 * 0.18425 \\ &= 6.467 + 5.865 \\ &= 12.3323 \end{aligned}$$

pv of liabilities
$$\begin{aligned} &= 10v^{10} + 20v^{15} \text{ at } 7\% \\ &= 10 * 0.50835 + 20 * 0.36245 \\ &= 5.0835 + 7.249 \\ &= 12.3324 \end{aligned}$$

Allowing for rounding, Redington's first condition is satisfied.

Redington's second condition states that the DMT of the assets should equal the DMT of the liabilities. Given denominator of DMTs of assets and liabilities have been shown to be equal, we only need to consider the numerators.

Numerator of DMT of assets
$$\begin{aligned} &= 7.404 * 2 * v^2 + 31.834 * 25 * v^{25} \text{ at } 7\% \\ &= 6.467 * 2 + 5.865 * 25 \\ &= 159.569 \end{aligned}$$

Numerator of DMT of liabilities $= 10*10*v^{10} + 20*15*v^{15}$ at 7%
$= 5.0835*10 + 7.249*15$
$= 159.569$

Allowing for rounding, Redington's 2nd condition is satisfied.

(ii) Profit $= 7.404v^2 + 31.834v^{25} - 10v^{10} - 20v^{15}$ at 7.5%
$= 6.40692 + 5.22011 - 4.85194 - 6.75932$
$= 0.015772$ i.e. a profit of £15,772

(iii) It can be seen that the spread of the assets is greater than the spread of the liabilities. This will mean that Redington's third condition for immunization is also satisfied, and that therefore a profit will occur if there is a small change in the rate of interest. Hence we would have anticipated a profit in (ii).

Parts (i) was answered well. Equating volatilities instead of DMTs was perfectly acceptable in this part. Part (ii) was also generally answered well although some candidates estimated the answer by using an estimation based on volatility rather than calculating the answer directly as asked. Part (iii) was less well answered with some candidates ignoring this part completely and others stating that Redington's 3rd condition was satisfied without further explanation.

7 Let K_t and S_t denote the forward price of the contract at time t, and the stock price at time t respectively.

Let r be the risk-free rate per annum at time $t = \frac{1}{2}$

Then, $K_0 = S_0 e^{0.04}$

and $K_{\frac{1}{2}} = 0.98\ S_0\ e^{\frac{1}{2}r}$

The value of the contract $V_{\frac{1}{2}}$ is $\left(K_{\frac{1}{2}} - K_0\right)e^{-\frac{1}{2}r}$

Hence $V_{\frac{1}{2}} = \left(K_{\frac{1}{2}} - K_0\right)e^{-\frac{1}{2}r}$

$= S_0 \times \left(0.98e^{\frac{1}{2}r} - e^{0.04}\right)e^{-\frac{1}{2}r}$

And

$V_{\frac{1}{2}} > 0$ when $0.98e^{\frac{1}{2}r} > e^{0.04}$

which is when $r > 2\ln\left(\dfrac{e^{0.04}}{0.98}\right) = 12.041\%$ p.a.

One of the worst answered questions on the paper. Some candidates, who did not complete the question, lost some of the marks that would have been available to them by not showing clear working e.g. writing down one half of a formula without explaining what the formula was supposed to represent.

8 (i) Let DPP be t. We want (all figures in £000s)

$$50{,}000 = 6{,}000a_{\overline{t}|}^{(2)} \quad \text{at 9\% p.a.}$$

$$= 6{,}000 \times \frac{i}{i^{(2)}} \times a_{\overline{t}|}$$

$$\Rightarrow a_{\overline{t}|} = \frac{50}{6 \times 1.022015}$$

$$= 8.1538268$$

$$\Rightarrow v^t = 1 - 8.1538268 \times 0.09$$

$$\Rightarrow t = \frac{\ln(1-8.1538268\times 0.09)}{\ln 1.09^{-1}} = 15.360$$

$\therefore$ Take DPP as 15.5 years

(ii) Profit at the end of 20 years is

$$-50{,}000 \times (1.09)^{15.5} \times (1.07)^{4.5} + 6{,}000 \times s_{\overline{15.5}|}^{(2)} \times (1.07)^{4.5} + X$$

where

$$s_{\overline{15.5}|}^{(2)} = \frac{(1+i)^{15.5}-1}{i^{(2)}} \text{ at 9\%}$$

$$= \frac{1.09^{15.5}-1}{0.088061}$$

$$= 31.8285476$$

and to find X we work in half-years:

$$X = 3{,}000 s_{\overline{9}|} \text{ at } j\% \text{ where } (1+j)^2 = 1.07$$

$$= 3{,}000 \times \frac{(1+j)^9 - 1}{j}$$

$$= 3{,}000 \times \frac{(1.07)^{9/2} - 1}{(1.07)^{1/2} - 1}$$

$$= 31{,}030.35528$$

$$\therefore \text{ Profit } = -257{,}814.7272 + 258{,}937.5717 + 31{,}030.35528$$

$$= 32{,}153.20$$

$$(= £32{,}153{,}200)$$

Part (i) was answered well although candidates lost marks for not recognising that the DPP could only be at the time of income receipt i.e. at the end of a half-year. Part (ii) was answered badly with some candidates ignoring the initial profit obtained at the end of the DPP. A common error in the calculation of the profit arising after the DPP was to calculate the present value rather than the accumulated value.

9 (i) We can find the one-year forward rates $f_{1,1}$ and $f_{2,1}$ from the spot rates y_1, y_2 and y_3:

$$(1+y_2)^2 = (1+y_1)(1+f_{1,1})$$

$$\Rightarrow (1+0.037)^2 = (1+0.036)(1+f_{1,1})$$

$$\Rightarrow f_{1,1} = 3.800\% \text{ p.a.}$$

and

$$(1+y_3)^3 = (1+y_2)^2 (1+f_{2,1})$$

$$\Rightarrow (1.038)^3 = (1.037)^2 (1+f_{2,1})$$

$$\Rightarrow f_{2,1} = 4.000\,\% \text{ p.a.}$$

(ii) (a) Price per £100 nominal

$$= 4\left(\underset{3.6\%}{v} + \underset{3.7\%}{v^2} + \underset{3.8\%}{v^3}\right) + 105\underset{3.8\%}{v^3}$$

$$= 4 \times 2.78931 + 105 \times 0.89414$$

$$= £105.0425$$

(b) Let yc_2 = two-year par yield

$$1 = yc_2\left(\underset{3.6\%}{v} + \underset{3.7\%}{v^2}\right) + \underset{3.7\%}{v^2}$$

$$\Rightarrow yc_2 = 3.6982\ \%\ \text{p.a.}$$

Questions on the term structure of interest rates have caused significant problems for candidates in past years but this question was generally answered very well.

10 (i) Let X = initial payment

$$20000 = (X - 50)a_{\overline{25}|} + 50(Ia)_{\overline{25}|}$$

$$= (X - 50) \times 10.6748 + 50 \times 98.4789$$

$$= 10.6748X - 533.74 + 4923.95$$

$$\Rightarrow X = \frac{15609.80}{10.6748} = £1,462.31.$$

(ii) After 3 years, capital o/s is:

$$1562.31a_{\overline{22}|} + 50(Ia)_{\overline{22}|}$$

$$= 1562.31 \times 10.2007 + 50 \times 87.1264$$

$$= £\,20,293.01$$

(iii) The loan has actually increased from £20,000 to £20,293.01. The reason for this is that the loan is being repaid by an increasing annuity and, in the early years, the interest is not covered by the repayments (e.g. 1st year: Interest is 0.08 × 20000 = £1,600 but 1st instalment is £1462.31 and so interest is not covered).

(iv) Total of instalments paid

$$= 25\times1462.31+\frac{24\times25}{2}\times50=51557.66$$

$\Rightarrow$ Total interest = $51557.66-20000$ = £31557.66

Parts (i) and (ii) were answered well, although in part (ii) some candidates incorrectly calculated the instalment that would be paid in the fourth year. Part (iii) was also answered relatively better than similar explanation questions in previous years. Many candidates failed to include the effect of the increasing payments in the calculation of the total instalments in part (iv) despite having correctly allowed for this in earlier parts.

11 (i) $PV=\int_4^{10}3{,}000\nu(t)\,dt$

where $\nu(t)$ is as follows:

$0\le t<4$

$$\nu(t)=e^{-\int_{o}^{t}(0.03+0.01t)dt}\quad=e^{-\left[0.03t+\frac{1}{2}x0.01t^2\right]}$$

$4\le t<6$

$$\nu(t)\ =e^{-0.20}.e^{-\int_4^t0.07dt}=e^{-0.20}.e^{(-0.07t+0.28)}$$

$$=e^{0.08-0.07t}$$

$t\ge6$

$$\nu(t)\ =e^{-0.34}.e^{-\int_6^t0.09dt}=e^{-0.34}.e^{(-0.09t+0.54)}$$

$$=e^{(0.20-0.09t)}$$

$$\Rightarrow PV\ =3{,}000\int_4^6\left(e^{0.08-0.07t}\right)dt\ +3{,}000\int_6^{10}e^{(0.20-0.09t)}dt$$

$$=\frac{3{,}000e^{0.08}}{-0.07}\left[e^{-0.42}-e^{-0.28}\right]+\frac{3{,}000e^{0.20}}{-0.09}\left[e^{-0.90}-e^{-0.54}\right]$$

$$=4584.02+7172.83=\$11{,}756.85$$

(ii) $11.75685 = 3\left(\bar{a}_{\overline{10|}} - \bar{a}_{\overline{4|}}\right)$

at i = 6%, RHS = 3(1.029709)[7.3601 − 3.4651] = 12.03215

at i = 7%, RHS = 3(1.034605)[7.0236 − 3.3872] = 11.28671

by interpolation

$$\therefore i = 0.06 + \left(\frac{12.03215 - 11.75685}{12.03215 - 11.28671} \times 0.01\right) = 0.06369 \text{ i.e. } 6.4\%$$

(actual answer is 6.36%)

One of the worst answered questions on the paper with the different formulation of a question based on varying forces of interest causing problems for many candidates. It is also possible to answer part (i) as a combination of continuous deferred annuities. Part (ii) was poorly answered even by candidates who had made a good attempt to part (i).

12 (i) $1 + i_t \sim \text{LogNormal}\left(\mu, \sigma^2\right)$

$$S_{12} = \prod_{1}^{12}(1 + i_t)$$

$$\Rightarrow \ln S_{12} = \sum_{1}^{12} \ln(1 + i_t) \sim N\left(12\mu, 12\sigma^2\right)$$

$$E(1 + i_t) = 1.08 \quad = \exp\left\{\mu + \sigma^2 / 2\right\}$$

$$\text{Var}(1 + i_t) = 0.05^2 \quad = \exp\left\{2\mu + \sigma^2\right\}\left(\exp\left(\sigma^2\right) - 1\right)$$

$$= 1.08^2\left(\exp\left(\sigma^2\right) - 1\right)$$

$$\Rightarrow e^{\sigma^2} \quad = 1 + \frac{0.05^2}{1.08^2}$$

$$\Rightarrow \sigma^2 \quad = 0.002141053$$

$$\mu \quad = \ln 1.08 - \sigma^2 / 2$$

$$= 0.075890514$$

Hence S_{12} has LogNormal distribution with parameters 0.910686 and 0.025692636

(ii) *PV* of annuity at time 12:

$$PV = 4000\underbrace{\ddot{a}_{\overline{4}|}^{(12)}}_{7\%} + 5000\underbrace{\ddot{a}_{\overline{2}|}^{(4)}v^4}_{7\%} + 5000\underbrace{\ddot{a}_{\overline{2}|}^{(4)}}_{9\%}\underbrace{v^6}_{7\%} + 6000\underbrace{\bar{a}_{\overline{4}|}v^2}_{9\%}\underbrace{v^6}_{7\%}$$

$$\begin{aligned} &= 1000\,(4\times1.037525\times3.3872+5\times1.043380\times1.8080\times0.76290 \\ &\qquad +5\times1.055644\times1.7591\times0.66634 \\ &\qquad +6\times1.044354\times3.2397\times0.84168\times0.66634) \end{aligned}$$

$$\begin{aligned} &= 1000\times(14.057219+7.195791+6.186911+11.385358) \\ &= 38,825.28 \end{aligned}$$

Hence

$$\begin{aligned} \text{Prob}\,(18,000S_{12} \geq 38,825.28) &= \text{Prob}\,(S_{12} \geq 2.15696) \\ &= \text{Prob}\left(Z \geq \frac{\ln(2.15696) - 0.910686}{\sqrt{0.025692636}}\right) \\ &= \text{Prob}\,(Z \geq -0.8858) \\ &= \Phi(0.89) \\ &= 0.81 \\ &\text{i.e. } 81\% \end{aligned}$$

This question provided the greatest range of quality of answers. Many candidates scored well on part (i) although common errors included assuming that $E(1+i_t) = 0.08$ *and/or that* $\text{Var}(1+i_t) = 0.05$. *Few candidates calculated the correct value of the required present value in part (ii) and candidates who made errors in this part lost further marks by not showing clear working or sufficient intermediate steps (although the examiners recognise that some candidates might have been under time pressure by the time they attempted this question). The probability calculation was often answered well by candidates who attempted this part.*

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

22 September 2014 (am)

Subject CT1 – Financial Mathematics
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a new page.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 Describe how cash flows are exchanged in an interest rate swap. [2]

2 A life insurance company is issuing a single premium policy which will pay out £200,000 in 20 years' time. The interest rate the company will earn on the invested fund throughout the 20 years will be 4% per annum effective with probability 0.25 or 7% per annum effective with probability 0.75. The insurance company uses the expected annual interest rate to determine the premium.

(i) Calculate the premium. [2]

(ii) Calculate the expected profit made by the insurance company at the end of the policy. [2]
[Total 4]

3 A 91-day treasury bill is bought for £98.83 and is redeemed at £100.

(i) Calculate the annual effective rate of interest from the bill. [3]

(ii) Calculate the annual equivalent simple rate of interest. [2]
[Total 5]

4 A fund had a value of £2.0 million on 1 January 2013. On 1 May 2013, £2.5 million was invested. Immediately before this investment, the value of the fund was £2.1 million. At the close of business on 31 December 2013, the value of the fund was £4.2 million.

(i) Calculate the annual effective time-weighted rate of return for 2013. [2]

(ii) Calculate the annual effective money-weighted rate of return for 2013. [3]

(iii) Comment on your answers to parts (i) and (ii). [2]
[Total 7]

5 Calculate, at a rate of interest of 5% per annum effective:

(i) $a_{\overline{5}|}^{(12)}$ [1]

(ii) ${}_{4|}a_{\overline{15}|}$ [1]

(iii) $(I\bar{a})_{\overline{10}|}$ [1]

(iv) $(\bar{I}\bar{a})_{\overline{10}|}$ [1]

(v) the present value of an annuity that is paid annually in advance for 10 years with a payment of 12 in the first year, 11 in the second year and thereafter reducing by 1 each year. [2]

[Total 6]

6 A Eurobond has been issued by a company that pays annual coupons of 5% per annum annually in arrear and is redeemable at par in exactly 10 years' time.

(i) Calculate the purchase price of the bond at issue at a rate of interest of 4% per annum effective assuming that tax is paid on the coupon payments at a rate of 20%. [2]

(ii) Calculate the discounted mean term of the bond at a rate of interest of 4% per annum effective, ignoring tax. [3]

(iii) (a) Explain why the discounted mean term of the gross payments from the bond is lower than the discounted mean term of the net payments.

(b) State two factors other than the size of the coupon payments that would affect the discounted mean term of the bond. [3]

(iv) Calculate the price of the bond three months after issue at a rate of interest of 4% per annum effective assuming tax is paid on the coupon payments at a rate of 20%. [1]

[Total 9]

7 The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t) = \begin{cases} 0.03 & \text{for} \quad 0 < t \le 10 \\ 0.003t & \text{for} \quad 10 < t \le 20 \\ 0.0001t^2 & \text{for} \quad t > 20 \end{cases}$$

(i) Calculate the present value of a unit sum of money due at time $t = 28$. [7]

(ii) (a) Calculate the equivalent constant force of interest from $t = 0$ to $t = 28$.

(b) Calculate the equivalent annual effective rate of discount from $t = 0$ to $t = 28$. [3]

A continuous payment stream is paid at the rate of $e^{-0.04t}$ per unit time between $t = 3$ and $t = 7$.

(iii) Calculate the present value of the payment stream. [4]

[Total 14]

8 (i) Explain what is meant by the following theories of the shape of the yield curve:

(a) market segmentation theory
(b) liquidity preference theory [4]

Short-term, one-year annual effective interest rates are currently 6%; they are expected to be 5% in one year's time; 4% in two years' time and 3% in three years' time.

(ii) Calculate the gross redemption yields from one-year, two-year, three–year and four-year zero coupon bonds using the above expected interest rates. [4]

The price of a coupon-paying bond is calculated by discounting individual payments from the bond at the zero–coupon yields in part (ii).

(iii) Calculate the gross redemption yield of a bond that pays a coupon of 4% per annum annually in arrear and is redeemed at 110% in exactly four years. [5]

(iv) Explain why the gross redemption yield of a bond that pays a coupon of 8% per annum annually in arrear and is redeemed at par would be greater than that calculated in part (iii). [2]

The government introduces regulations that require banks to hold more government bonds with very short terms to redemption.

(v) Explain, with reference to market segmentation theory, the likely effect of this regulation on the pattern of spot rates calculated in part (ii). [2]
[Total 17]

9 A government issued a number of index-linked bonds on 1 June 2012 which were redeemed on 1 June 2014. Each bond had a nominal coupon of 2% per annum, payable half yearly in arrear and a nominal redemption price of 100%. The actual coupon and redemption payments were indexed according to the increase in the retail price index between three months before the issue date and three months before the relevant payment dates. No adjustment is made to allow for the actual date of calculation of the price index within the month or the precise coupon payment date within the month.

The values of the retail price index in the relevant months were:

Date	*Retail Price Index*
March 2012	112
June 2012	113
September 2012	116
December 2012	117
March 2013	117
June 2013	118
September 2013	120
December 2013	121
March 2014	121
June 2014	122

An investor purchased £3.5m nominal of the bond at the issue date and held it until it was redeemed. The investor was subject to tax on coupon payments at a rate of 25%.

(i) Calculate the incoming net cash flows the investor received. [5]

(ii) Express the cash flows in terms of 1 June 2012 prices. [4]

(iii) Calculate the purchase price of the bond per £100 nominal if the real net redemption yield achieved by the investor was 1.5% per annum effective. [3]

When the investor purchased the security, he expected the retail price index to rise much more slowly than it did in practice.

(iv) Explain whether the investor’s expected net real rate of return at purchase would have been greater than 1.5% per annum effective. [2]

In September 2012, the government indicated that it might change the price index to which payments were linked to one which tends to rise more slowly than the retail price index.

(v) Explain the likely impact of such a change on the market price of index-linked bonds. [2]

[Total 16]

10 A student is considering whether to attend university or enter a profession immediately upon leaving school. If he enters the profession immediately, his salary is expected to be as follows.

Year 1: £15,000
Year 2: £18,000
Year 3: £20,000

In each subsequent year the expected salary would rise by 1% per annum compound. The salary is assumed to be received monthly in arrear for 40 years.

If he attends university, the fees and other costs will be £15,000 per annum for three years, paid annually in advance. After attending university, the student's potential earnings will rise. Immediately after leaving university, he expects to earn £22,000 in the first year, £25,000 in the second year and £28,000 in the third year. Thereafter, his salary is expected to rise each year by 1.5% per annum compound. The salary would be paid monthly in arrear for 37 years.

(i) Calculate the present value of the student's earnings if he enters the profession immediately at a rate of interest of 7% per annum effective. [7]

(ii) Calculate the net present value of the decision to attend university at a rate of interest of 7% per annum effective and hence determine whether attending university would be a more attractive option. [9]

(iii) Explain why attending university would be relatively more attractive at lower interest rates. [2]

The student wishes to consider the effect of taxation on earnings.

(iv) Determine the rate of income tax above which the option of attending university would be less attractive financially than that of entering the profession immediately. [2]

[Total 20]

END OF PAPER

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2014 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

November 2014

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2014 paper

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates. In general the non-numerical questions were answered poorly by marginal candidates. This applied to bookwork questions such as Q1 and Q8(i) as well as questions requiring interpretation of answers such as Q4(iii), Q8(iv) and (v) and Q9(iv) and (v).

1 One party agrees to pay to the other a regular series of fixed amounts for a certain term. In exchange, the second party agrees to pay a series of variable amounts in the same currency based on the level of a short term interest rate. [2]

2 (i) Expected annual interest rate

$= 0.25 \times 4 + 0.75 \times 7 = 6.25\%$

Let premium = P

$P\,(1.0625)^{20} = 200{,}000$

$\therefore\ P = £59{,}490.99$ [2]

(ii) Expected accumulation is:

$59{,}490.99\,(0.25 \times 1.04^{20} + 0.75 \times 1.07^{20})$

$= 205{,}246.55$

$\therefore$ Expected profit = £5,246.55 [2]

[Total 4]

Many candidates struggled to distinguish between the use of an expected annual interest rate and the expected accumulation after 20 years.

3 (i) $98.83 = 100\,(1 + i)^{-91/365}$

$$\ln(1+i) = \left(-\frac{365}{91}\right) \times \ln(98.83/100) = 0.047205$$

Therefore $i = 0.04834$. [3]

(ii) The rate of interest over 91 days is

$(100 - 98.83) / 98.83 = 0.011839$

The simple rate per annum is:

$$0.011839 \times \frac{365}{91} = 0.04748$$ [2]

[Total 5]

4 (i) Let i be the TWRR per annum effective, then:

$$1+i = \frac{2.1}{2.0} \times \frac{4.2}{2.1+2.5} = 0.95870$$

$\Rightarrow$ TWRR $= -4.130\%$p.a. [2]

(ii) Let i be the MWRR per annum effective, then:

$$2.0(1+i) + 2.5(1+i)^{2/3} = 4.2$$

Try: -8% LHS = 4.20482
-9% LHS = 4.16765

Then $-i = 0.08 + 0.01 \times \left(\frac{4.20482 - 4.2}{4.20482 - 4.16765} \right)$

$i \approx -8.130\% = -8.1\%$p.a. (Exact answer is -8.12985%) [3]

(iii) The MWRR is affected by the timing and amount of cashflows. The fund performs relatively worse when the size of the fund is largest and this will have a greater effect on the MWRR which is consequently lower than the TWRR. [2]

[Total 7]

Parts (i) and (ii) were well-answered. In part (iii), examiners were looking for specific comments regarding this scenario and not just a statement of the bookwork.

5 (i) $\frac{i}{i^{(12)}} a_{\overline{5}|} = 1.022715 \times 4.3295 = 4.4278$ [1]

(ii) $a_{\overline{19}|} - a_{\overline{4}|} = 12.0853 - 3.5460$ [1]

$= 8.5394$

(iii) $\frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{\delta}$

$= \frac{1.05 \times 7.7217 - 10 \times 0.61391}{0.04879}$

$= 40.3501$ [1]

(iv) $\dfrac{\bar{a}_{\overline{10|}} - 10v^{10}}{\delta} = \dfrac{1.024797 \times 7.7217 - 10 \times 0.61391}{0.04879}$

$= 36.3613.$ [1]

(v) Present value is:

$13\ddot{a}_{\overline{10|}} - (I\ddot{a})_{\overline{10|}}$

$= 1.05 \times (13 \times 7.7217 - 39.3738)$

$= 64.0592$ [2]

[Total 6]

Generally well-answered although some candidates were unable to distinguish between the increasing annuities in parts (iii) and (iv).

6 (i) $P = 0.8 \times 5\, a_{\overline{10|}} + 100v^{10}$ @ 4% per annum.

$a_{\overline{10|}} = 8.1109 \qquad v^{10} = 0.67556$

$P = 0.8 \times 5 \times 8.1109 + 100 \times 0.67556 = 100$

No loss of marks for general reasoning. [2]

(ii) DMT $= \dfrac{\sum tC_t\, v^t}{\sum C_t v^t}$

$= \dfrac{5(Ia)_{\overline{10|}} + 10 \times 100v^{10}}{5a_{\overline{10|}} + 100v^{10}}$

$= \dfrac{5 \times 41.9922 + 10 \times 100 \times 0.67556}{5 \times 8.1109 + 100 \times 0.67556}$

$= \dfrac{885.525}{108.1108} = 8.19$ years [3]

(iii) (a) On average the gross cash flows are earlier because of the higher coupon payments. Therefore the discounted mean term would be lower.

(b) Term of the bond
The gross redemption yield/interest rate at which payments are discounted.

[3]

(iv) All payments are 3 months closer. Therefore, purchase price would be

$$100(1.04)^{1/4} = 100.9853$$ [1]

[Total 9]

Parts (i) and (ii) were answered well. Many candidates' arguments in part (iii)(a) were unclear.

7 (i)

$$A(0,10) = e^{\int_0^{10} 0.03\,dt} = e^{[0.03t]_0^{10}} = e^{0.3}$$

$$A(10,20) = e^{\int_{10}^{20} 0.003t\,dt}$$

$$= e^{\left[\frac{0.003t^2}{2}\right]_{10}^{20}}$$

$$= e^{(0.6-0.15)} = e^{0.45}$$

$$A(20,28) = e^{\int_{20}^{28} 0.0001t^2\,dt}$$

$$= e^{\left[\frac{0.0001t^3}{3}\right]_{20}^{28}} = e^{0.73173-0.26666} = e^{0.46507}$$

Required PV

$$= \frac{1}{A(0,10)\,A(10,20)\,A(20,28)} = e^{-0.3-0.45-0.46507} = e^{-1.21507}$$

$$= 0.29669$$ [7]

(ii) (a) $0.29669 = e^{-28\delta}$

$$\frac{\ln 0.29669}{-28} = \delta = 0.04340 = 4.340\% \text{ per annum}$$

(b) $(1-d)^{28} = 0.29669$

$1 - d = 0.95753$

$d = 0.04247 = 4.247\%$ per annum [3]

(iii) $v(t) = e^{-\int_0^t 0.03 ds} = e^{-0.03t}$

$\rho(t) = e^{-0.04t}$

We require:

$$\int_3^7 e^{-0.03t} e^{-0.04t} dt = \int_3^7 e^{-0.07t} dt$$

$$= \left[\frac{-e^{-0.07t}}{0.07} \right]_3^7$$

$= -8.75181 + 11.57977$

$= 2.82797$ [4]

[Total 14]

Answered well. The common mistake was to calculate the effective rate of interest rather than the effective rate of discount in part (ii)(b).

8 (i) (a) Bonds of different terms are attractive to different investors, who will choose assets that are appropriate for their liabilities. The shape of the yield curve is determined by supply and demand at different terms to redemption.

(b) Longer dated bonds are more sensitive to interest rate movements than short dated bonds. It is assumed that risk averse investors will require compensation (in the form of higher yields) for the greater risk of loss on longer bonds.

[4]

(ii) Let i_t be the spot yield over t years.

One year: yield is 6% therefore $i_1 = 0.06$
Two years: $(1 + i_2)^2 = 1.06 \times 1.05$ therefore $i_2 = 0.054988$
Three years: $(1 + i_3)^3 = 1.06 \times 1.05 \times 1.04$ therefore $i_3 = 0.049968$.
Four years: $(1 + i_4)^4 = 1.06 \times 1.05 \times 1.04 \times 1.03$ therefore $i_4 = 0.04494$.

[4]

(iii) Price of bond is:

$4[(1.06)^{-1} + (1.054988)^{-2} + (1.049968)^{-3} + (1.04494)^{-4}]$
$\quad + 110 \times 1.04494^{-4}$
$= 4 \times 3.54454 + 92.26294$
$= 106.4411$

Find the gross redemption yield from:

$$106.4411 = 4a_{\overline{4|}} + 110v^4$$

Try 4%

$$a_{\overline{4|}} = 3.6299 \quad v^4 = 0.85480$$

RHS = 108.5476

Try 5%

$$a_{\overline{4|}} = 3.5460 \quad v^4 = 0.82270$$

RHS = 104.681

Interpolate between 4% and 5%:

$$i = 0.04 + 0.01 \times \frac{108.5476 - 106.4411}{108.5476 - 104.681}$$

$= 0.0454$

$= 4.54\%$

[5]

(iv) On average, the payments would be received earlier and discounted at higher spot rates. This means that the gross redemption yield (which is a weighted average of the interest rates used to discount the payments) would be higher.
[2]

(v) The earlier spot rates are likely to fall as a result of greater demand for the bonds with shorter terms to redemption. [2]
[Total 17]

Parts (i) and (iii) were generally answered well with correct approaches in part (iii) given full credit even if the calculations in part (ii) had been incorrect. In common with other similar questions on this paper, the reasoning questions in parts (iv) and (v) were poorly answered.

9 (i)

Date	*Nominal Cash Flow £m*	*Indexed Cash Flow £m*
1/12/2012	0.0075×3.5 = 0.02625	(116/112)×0.02625 = 0.0271875
1/6/2013	0.0075×3.5 = 0.02625	(117/112)×0.02625 = 0.0274219
1/12/2013	0.0075×3.5 = 0.02625	(120/112)×0.02625 = 0.028125
1/6/2014	(1 + 0.0075)×3.5 = 3.52625	(121/112)×3.52625 = 3.8096094

[5]

(ii)

Date	*Indexed Cash Flow £m*	*Index Ratio*	*Real Value of Cash flow £m*
1/12/2012	0.0271875	113/117	0.0262580
1/6/2013	0.0274219	113/118	0.0262599
1/12/2013	0.028125	113/121	0.0262655
1/6/2014	3.8096094	113/122	3.5285726

[4]

(iii) Value of £3.5m nominal is:

$$0.0262580v^{\frac{1}{2}} + 0.0262599v + 0.0262655v^{1\frac{1}{2}} + 3.5285726v^2$$

$$= 0.0262580 \times 0.992583 + 0.0262599 \times 0.98522 + 0.0262655 \times 0.97791$$

$$+ 3.5285726 \times 0.97066$$

$$= £3.502657\text{m}$$

$$\text{Per £100 nominal } = \frac{3.502671}{3.5} \times 100$$

$$= £100.0763$$ [3]

(iv) The expected rate of return at issue is likely to have been higher. Although the investor is compensated for the higher-than-expected inflation, the time lag used for indexation is likely to mean that he is not fully compensated. Therefore the actual real value of the cash flows is less than the expected real value of the cash flows at issue. [2]

(v) It is likely that the price will fall. The expected real value of the cash flows measured will be lower because the cash flows will be linked to an index expected to rise at a lower rate. [2]

[Total 16]

The most poorly answered question on the paper. Better candidates took advantage of the relatively large number of marks available in parts (i) and (ii) for straightforward calculation work. The important point in part (iii) is to note that the real redemption yield *equation uses inflation adjusted cashflows (in terms of 1 June 2012 prices in this case). In part (iv), the important point is that the time lag causes the investor not to be fully protected against inflation. If there had been no time lag, the actual increase in the retail price index would have no effect on the investor's real rate of return.*

10 (i) Present value of earning if university is not attended:

$$15,000\ a_{\overline{1}|}^{(12)} + 18,000\ a_{\overline{1}|}^{(12)} v + 20,000\ a_{\overline{1}|}^{(12)} v^2 + 20,000 a_{\overline{1}|}^{(12)}$$

$$\times 1.01 v^3 (1 + 1.01v + ... + 1.01^{36} v^{36})$$

$$= \frac{i}{i^{(12)}} a_{\overline{1}|} (15,000 + 18,000v + 20,000v^2)$$

$$+ \left(20,000 \frac{i}{i^{(12)}} a_{\overline{1}|} * 1.01 v^3 \right) \left(\frac{1 - 1.01^{37} v^{37}}{1 - 1.01v} \right)$$

$$\frac{i}{i^{(12)}} = 1.031691; \ a_{\overline{1}|} = v = 0.93458$$

$v^2 = 0.87344$; $v^3 = 0.81630$; $1.01^{37} = 1.445076$

$v^{37} = 0.08181.$

1.031691 × 0.93458 (15,000 + 18,000 × 0.93458 + 20,000 × 0.8734)

+ (20,000 × 1.031691 × 0.93458

$$\times 1.01 \times 0.81630) \ \left(\frac{1 - 1.445076 \times 0.081809}{1 - 1.01 \times 0.9346} \right)$$

= 47,527.46 + 15,898.86 × 15.7252

= £297,537.30 [7]

(ii) The cost of going to university is:

$$15,000 \times \ddot{a}_{\overline{3}|} @ 7\% = 42,120.27$$

$$\ddot{a}_{\overline{3}|} = 2.6243 \times 1.07 = 2.8080$$

The PV of the salary from attending university at the time of leaving university is:

$$22{,}000\, a_{\overline{1|}}^{(12)} + 25{,}000\, {}_{\overline{1|}}^{(12)} v + 28{,}000\, a_{\overline{1|}}^{(12)} v^2$$

$$+28{,}000 \quad a_{\overline{1|}}^{(12)} \times 1.015 v^3 (1 + 1.015 v + \ldots + 1.015^{33} v^{33})$$

$$= \frac{i}{i^{(12)}} a_{\overline{1|}} (22{,}000 + 25{,}000 v + 28{,}000 v^2)$$

$$+ \left(28{,}000 \frac{i}{i^{(12)}} a_{\overline{1|}} 1.015 v^3 \right) \left(\frac{1 - 1.015^{34} v^{34}}{1 - 1.015 v} \right)$$

$1.015^{34} = 1.658996$

$v^{34} = 0.10022$

$= 1.031691 \times 0.93458(22{,}000 + 25{,}000 \times 0.93458$

$+ 28{,}000 \times 0.87344) + 28{,}000 \times 1.031691 \times 0.93458 \times 1.015$

$\times\ 0.81630 \left(\dfrac{1 - 1.658996 \times 0.10022}{1 - 1.015 \times 0.93458} \right)$

$= 67{,}321.02 + 22{,}368.60 \times 16.21996$

$= 430{,}138.80$

PV at time of decision $= 430{,}138.80 \times v^3$

$= 430{,}138.80 \times 0.81630 =$ £351,121.46

There are various ways in which the answer can be rationalised.

NPV of benefit of going to university (net of earnings lost through the alternative course of action)

$= 351{,}121.46 - 42{,}120.27 - 297{,}537.30$

= £11,463.89 [9]

(iii) The costs of going to university are incurred earlier and the benefits received later. If the rate of interest is lower, then any loans taken out to finance attendance at university will be repaid more easily at a lower interest cost (answer could say that value of payments received later will rise by more when the interest rate falls). [2]

(iv) Tax is paid on income only at rate t.

Therefore, equation of value is:

$351{,}121.46\,(1-t) = 42{,}120.27 + 297{,}537.30 \times (1-t)$

$351{,}121.46 - 42{,}120.27 - 297{,}537.30$

$= t\,(351{,}121.46 - 297{,}537.30)$

$\therefore\ 11{,}463.89 = 53{,}584.16t$

$\therefore\ t = 0.2139$ or 21.39% [2]

[Total 20]

Parts (i) and (ii) were often well-answered although marginal candidates would have benefited from setting out their working more clearly and some candidates failed to' determine whether attending university would be a more attractive option' despite having completed the requisite calculations.

Part (iii) was poorly answered by marginal candidates with few such candidates correctly considering the relative timing of the costs and benefits. Few such candidates attempted part (iv) perhaps because of time pressure. Stronger candidates, however, often obtained close to full marks on the question.

END OF EXAMINERS' REPORT